

# DYNAMICS OF THE UNIVERSAL AREA-PRESERVING MAP ASSOCIATED WITH PERIOD DOUBLING: STABLE SETS

DENIS GAIDASHEV, TOMAS JOHNSON

**ABSTRACT.** It is known that the famous Feigenbaum-Coulet-Tresser period doubling universality has a counterpart for area-preserving maps of  $\mathbb{R}^2$ . A renormalization approach has been used in (Eckmann *et al* 1982) and (Eckmann *et al* 1984) in a computer-assisted proof of existence of a “universal” area-preserving map  $F_*$  — a map with orbits of all binary periods  $2^k, k \in \mathbb{N}$ . In this paper, we consider *infinitely renormalizable* maps — maps on the renormalization stable manifold in some neighborhood of  $F_*$  — and study their dynamics.

For all such infinitely renormalizable maps in a neighborhood of the fixed point  $F_*$  we prove the existence of a “stable” invariant Cantor set  $\mathcal{C}_F^\infty$  such that the Lyapunov exponents of  $F|_{\mathcal{C}_F^\infty}$  are zero, and whose Hausdorff dimension satisfies

$$\dim_H(\mathcal{C}_F^\infty) < 0.5324.$$

We also show that there exists a submanifold,  $\mathbf{W}_\omega$ , of finite codimension in the renormalization local stable manifold, such that for all  $F \in \mathbf{W}_\omega$  the set  $\mathcal{C}_F^\infty$  is “weakly rigid”: the dynamics of any two maps in this submanifold, restricted to the stable set  $\mathcal{C}_F^\infty$ , is conjugated by a bi-Lipschitz transformation that preserves the Hausdorff dimension.

## CONTENTS

1. Introduction	1
2. Renormalization for area-preserving reversible maps	3
3. Hyperbolic sets for maps in $\mathbf{F}_*$	7
4. Statement of results	9
5. Some notation and definitions	10
6. A stable invariant set as a Hausdorff limit of hyperbolic sets	11
7. The stable set as a Cantor set	22
8. “Weak” rigidity	24
9. Some concluding remarks	29
10. Acknowledgments	30
References	30

## 1. INTRODUCTION

Universality — independence of the quantifiers of the geometry of orbits and bifurcation cascades in families of maps of the choice of a particular family — has

---

2000 *Mathematics Subject Classification.* 37E20, 37F25, 37D05, 37D20, 37C29, 37A05, 37G15, 37M99.

been abundantly observed in area-preserving maps, both as the period-doubling universality (Derrida and Pomeau 1980, Helleman 1980, Benettin *et al* 1980, Collet *et al* 1981, Eckmann *et al* 1982, Eckmann *et al* 1984, Gaidashev and Koch 2008) and as the universality associated with the break-up of invariant surfaces (Shenker and Kadanoff 1982, MacKay 1982, MacKay 1983, Mehr and Escande 1984), and in Hamiltonian flows (Escande and Doveil 1981, Abad *et al* 2000, Abad *et al* 1998, Koch 2002, Koch 2004, Koch 2008, Gaidashev and Koch 2004, Gaidashev 2005, Kocić 2005).

To prove universality one usually introduces a *renormalization* operator on a functional space, and demonstrates that this operator has a *hyperbolic fixed point*. The renormalization approach to universality has been very successful in one-dimensional dynamics, and has led to explanation of universality in unimodal maps (Epstein 1986, Epstein 1989, Lyubich 1999), critical circle maps (de Faria 1992, de Faria 1999, Yampolsky 2002, Yampolsky 2003) and holomorphic maps with a Siegel disk (McMullen 1998, Yampolsky 2007, Gaidashev and Yampolsky 2007). There is, however, at present no deep understanding of universality in conservative systems, other than in the “trivial” case of the universality for systems “near integrability” (Koch 2002, Koch 2004, Gaidashev 2005, Kocić 2005, Khanin *et al* 2007).

It is worth noting that universality in conservative systems seems to be completely different from that in one-dimensional and dissipative maps. As it has been shown in (Collet *et al* 1980, de Carvalho *et al* 2005, Lyubich and Martens 2008), the case of very dissipative systems is largely reducible to the one-dimensional Feigenbaum-Coullet-Tresser universality.

For families of area-preserving maps a universal infinite period-doubling cascade was observed by several authors in the early 80’s (Derrida and Pomeau 1980, Helleman 1980, Benettin *et al* 1980, Bountis 1981, Collet *et al* 1981). The existence of a hyperbolic fixed point for the period-doubling renormalization operator has been proved with computer-assistance in (Eckmann *et al* 1984).

In (Gaidashev and Johnson 2009) we used the method of covering relations (see, e.g. (Zgliczyński 1997, Zgliczyński and Gidea 2004, Kokubu *et al* 2007, Zgliczyński 2009, CAPD 2009)) in rigorous computations to construct hyperbolic sets for all maps in some neighborhood of the fixed point of the renormalization operator. The Hausdorff dimension of these hyperbolic sets has been estimated with the help of the Duarte Distortion Theorem (see, e.g. (Duarte 2000)) which enables one to use the distortion of a Cantor set to find bounds on its dimension.

In this paper, we prove that *infinitely renormalizable* maps in a neighborhood of existence of the hyperbolic sets also admit a “stable” set. This set is a bounded invariant set, such that the maximal Lyapunov exponent at any point of this set is zero. Together with our result from (Gaidashev and Johnson 2009), this demonstrates that for all reversible area-preserving infinitely renormalizable maps in some neighborhood of the renormalization fixed point, there are coexisting hyperbolic and stable sets.

We also address the issues of rigidity of the stable set and invariance of its Hausdorff dimension. Similar issues have been investigated in (de Carvalho *et al* 2005) for attractors of very dissipative two-dimensional maps, where it has been shown that the regularity of conjugacy of attractors for two infinitely renormalizable

maps  $F$  and  $\tilde{F}$  has a definite upper bound

$$(1) \quad \alpha \leq \frac{1}{2} \left( 1 + \min \left\{ \frac{\log \text{Jac}(F)}{\log \text{Jac}(\tilde{F})}, \frac{\log \text{Jac}(\tilde{F})}{\log \text{Jac}(F)} \right\} \right),$$

where  $\text{Jac}(F)$  is the “average” Jacobian of the map  $F$ . The authors of (de Carvalho *et al* 2005) put forward two questions: 1) whether the Hausdorff dimension of the attractor of an infinitely renormalizable map depends only on its average Jacobian, and 2) how regular is the conjugacy when  $\text{Jac}(F) = \text{Jac}(\tilde{F})$ . In this regard, we obtain a partial result along similar lines in the “extreme” case of area-preserving maps (constant Jacobian equal to one): we prove that there exists a subset of locally infinitely renormalizable maps such the actions of any two maps from this subset on their stable sets are conjugate by a bi-Lipschitz map which preserves the Hausdorff dimension.

We can not make a definite conclusion about whether this subset is equal to the whole set of locally infinitely renormalizable maps, or strictly smaller, because a sharp bound on the convergence rate of renormalizations of infinitely renormalizable maps is not known to date.

Finally, we provide an upper bound on the Hausdorff dimension of the stable set for all infinitely renormalizable maps.

## 2. RENORMALIZATION FOR AREA-PRESERVING REVERSIBLE MAPS

An “area-preserving map” will mean an exact symplectic diffeomorphism of a subset of  $\mathbb{R}^2$  onto its image.

Recall, that an area-preserving map can be uniquely specified by its generating function  $S$ :

$$(2) \quad \begin{pmatrix} x \\ -S_1(x, y) \end{pmatrix} \xrightarrow{F} \begin{pmatrix} y \\ S_2(x, y) \end{pmatrix}, \quad S_i \equiv \partial_i S.$$

Furthermore, we will assume that  $F$  is reversible, that is

$$(3) \quad T \circ F \circ T = F^{-1}, \quad \text{where} \quad T(x, u) = (x, -u).$$

For such maps it follows from (2) that

$$S_1(y, x) = S_2(x, y) \equiv s(x, y),$$

and

$$(4) \quad \begin{pmatrix} x \\ -s(y, x) \end{pmatrix} \xrightarrow{F} \begin{pmatrix} y \\ s(x, y) \end{pmatrix}.$$

It is this “little”  $s$  that will be referred to below as “the generating function”. If the equation  $-s(y, x) = u$  has a unique differentiable solution  $y = y(x, u)$ , then the derivative of such a map  $F$  is given by the following formula:

$$(5) \quad DF(x, u) = \begin{bmatrix} -\frac{s_2(y(x, u), x)}{s_1(y(x, u), x)} & -\frac{1}{s_1(y(x, u), x)} \\ s_1(x, y(x, u)) - s_2(x, y(x, u)) \frac{s_2(y(x, u), x)}{s_1(y(x, u), x)} & -\frac{s_2(x, y(x, u))}{s_1(y(x, u), x)} \end{bmatrix}.$$

The period-doubling phenomenon can be illustrated with the area-preserving Hénon family (cf. (Bountis 1981)) :

$$H_a(x, u) = (-u + 1 - ax^2, x).$$

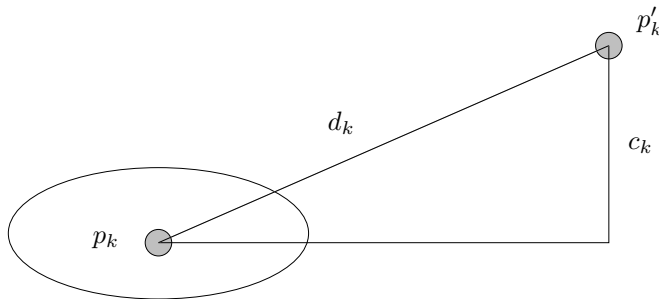


FIGURE 1. The geometry of the period doubling.  $p_k$  is the further elliptic point that has bifurcated from the hyperbolic point  $p'_k$ .

Maps  $H_a$  have a fixed point  $((-1 + \sqrt{1+a})/a, (-1 + \sqrt{1+a})/a)$  which is stable for  $-1 < a < 3$ . When  $a_1 = 3$  this fixed point becomes unstable, at the same time an orbit of period two is born with  $H_a(x_{\pm}, x_{\mp}) = (x_{\mp}, x_{\pm})$ ,  $x_{\pm} = (1 \pm \sqrt{a-3})/a$ . This orbit, in turn, becomes unstable at  $a_2 = 4$ , giving birth to a period 4 stable orbit. Generally, there exists a sequence of parameter values  $a_k$ , at which the orbit of period  $2^{k-1}$  turns unstable, while at the same time a stable orbit of period  $2^k$  is born. The parameter values  $a_k$  accumulate on some  $a_{\infty}$ . The crucial observation is that the accumulation rate

$$(6) \quad \lim_{k \rightarrow \infty} \frac{a_k - a_{k-1}}{a_{k+1} - a_k} = 8.721\dots$$

is universal for a large class of families, not necessarily Hénon.

Furthermore, the  $2^k$  periodic orbits scale asymptotically with two scaling parameters

$$(7) \quad \lambda = -0.249\dots, \quad \mu = 0.061\dots$$

To explain how orbits scale with  $\lambda$  and  $\mu$  we will follow (Bountis 1981). Consider an interval  $(a_k, a_{k+1})$  of parameter values in a “typical” family  $F_a$ . For any value  $\alpha \in (a_k, a_{k+1})$  the map  $F_{\alpha}$  possesses a stable periodic orbit of period  $2^k$ . We fix some  $\alpha_k$  within the interval  $(a_k, a_{k+1})$  in some consistent way; for instance, by requiring that the restriction of  $F_{\alpha_k}^{2^k}$  to a neighborhood of a stable periodic point in the  $2^k$ -periodic orbit is conjugate, via a diffeomorphism  $H_k$ , to a rotation with some fixed rotation number  $r$ . Let  $p'_k$  be some unstable periodic point in the  $2^{k-1}$ -periodic orbit, and let  $p_k$  be the further of the two stable  $2^k$ -periodic points that bifurcated from  $p'_k$ . Denote with  $d_k = |p'_k - p_k|$ , the distance between  $p_k$  and  $p'_k$ . The new elliptic point  $p_k$  is surrounded by invariant ellipses; let  $c_k$  be the distance between  $p_k$  and  $p'_k$  in the direction of the minor semi-axis of an invariant ellipse surrounding  $p_k$ , see Figure 1. Then,

$$\frac{1}{\lambda} = - \lim_{k \rightarrow \infty} \frac{d_k}{d_{k+1}}, \quad \frac{\lambda}{\mu} = - \lim_{k \rightarrow \infty} \frac{\rho_k}{\rho_{k+1}}, \quad \frac{1}{\lambda^2} = \lim_{k \rightarrow \infty} \frac{c_k}{c_{k+1}},$$

where  $\rho_k$  is the ratio of the smaller and larger eigenvalues of  $DH_k(p_k)$ .

This universality can be explained rigorously if one shows that the *renormalization* operator

$$(8) \quad R[F] = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F,$$

where  $\Lambda_F$  is some  $F$ -dependent coordinate transformation, has a fixed point, and the derivative of this operator is hyperbolic at this fixed point.

It has been argued in (Collet *et al* 1981) that  $\Lambda_F$  is a diagonal linear transformation. Furthermore, such  $\Lambda_F$  has been used in (Eckmann *et al* 1982) and (Eckmann *et al* 1984) in a computer assisted proof of existence of a reversible renormalization fixed point  $F_*$  and hyperbolicity of the operator  $R$ .

We will now derive an equation for the generating function of the renormalized map  $\Lambda_F^{-1} \circ F \circ F \circ \Lambda_F$ .

Applying a reversible  $F$  twice we get

$$\begin{pmatrix} x' \\ -s(z', x') \end{pmatrix} \xrightarrow{F} \begin{pmatrix} z' \\ s(x', z') \end{pmatrix} = \begin{pmatrix} z' \\ -s(y', z') \end{pmatrix} \xrightarrow{F} \begin{pmatrix} y' \\ s(z', y') \end{pmatrix}.$$

According to (Collet *et al* 1981)  $\Lambda_F$  can be chosen to be a linear diagonal transformation:

$$\Lambda_F(x, u) = (\lambda x, \mu u).$$

We, therefore, set  $(x', y') = (\lambda x, \lambda y)$ ,  $z'(\lambda x, \lambda y) = z(x, y)$  to obtain:

$$(9) \quad \begin{pmatrix} x \\ -\frac{1}{\mu}s(z, \lambda x) \end{pmatrix} \xrightarrow{\Lambda_F} \begin{pmatrix} \lambda x \\ -s(z, \lambda x) \end{pmatrix} \xrightarrow{F \circ F} \begin{pmatrix} \lambda y \\ s(z, \lambda y) \end{pmatrix} \xrightarrow{\Lambda_F^{-1}} \begin{pmatrix} y \\ \frac{1}{\mu}s(z, \lambda y) \end{pmatrix},$$

where  $z(x, y)$  solves

$$(10) \quad s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)) = 0.$$

If the solution of (10) is unique, then  $z(x, y) = z(y, x)$ , and it follows from (9) that the generating function of the renormalized  $F$  is given by

$$(11) \quad \tilde{s}(x, y) = \mu^{-1}s(z(x, y), \lambda y).$$

One can fix a set of normalization conditions for  $\tilde{s}$  and  $z$  which serve to determine scalings  $\lambda$  and  $\mu$  as functions of  $s$ . For example, the normalization  $s(1, 0) = 0$  is reproduced for  $\tilde{s}$  as long as  $z(1, 0) = z(0, 1) = 1$ . In particular, this implies that  $s(\lambda, 1) + s(0, 1) = 0$ . Furthermore, the condition  $\partial_1 s(1, 0) = 1$  is reproduced as long as  $\mu = \partial_1 z(1, 0)$ .

We will now summarize the above discussion in the following definition of the renormalization operator acting on generating functions originally due to the authors of (Eckmann *et al* 1982) and (Eckmann *et al* 1984):

**Definition 2.1.**

$$(12) \quad \mathcal{R}_{EKW}[s](x, y) = \mu^{-1}s(z(x, y), \lambda y),$$

where

$$(13) \quad 0 = s(\lambda x, z(x, y)) + s(\lambda y, z(x, y)),$$

$$(14) \quad 0 = s(\lambda, 1) + s(0, 1) \quad \text{and} \quad \mu = \partial_1 z(1, 0).$$

**Definition 2.2.** The Banach space of functions  $s(x, y) = \sum_{i,j=0}^{\infty} c_{ij} x^i y^j$ , analytic on a bi-disk

$$|x - 0.5| < \rho, |y - 0.5| < \rho,$$

for which the norm

$$\|s\|_{\rho} = \sum_{i,j=0}^{\infty} |c_{ij}| \rho^{i+j}$$

is finite, will be referred to as  $\mathcal{A}(\rho)$ .

$\mathcal{A}_s(\rho)$  will denote its symmetric subspace  $\{s \in \mathcal{A}(\rho) : s_1(x, y) = s_1(y, x)\}$ .

As we have already mentioned, the following has been proved with the help of a computer in (Eckmann *et al* 1982) and (Eckmann *et al* 1984):

**Theorem 1.** There exist a polynomial  $s_a \in \mathcal{A}_s(\rho)$  and a ball  $\mathcal{B}_r(s_a) \subset \mathcal{A}_s(\rho)$ ,  $r = 6.0 \times 10^{-7}$ ,  $\rho = 1.6$ , such that the operator  $\mathcal{R}_{EKW}$  is well-defined and analytic on  $\mathcal{B}_r(s_a)$ .

Furthermore, its derivative  $D\mathcal{R}_{EKW}|_{\mathcal{B}_r(s_a)}$  is a compact linear operator, and has exactly two eigenvalues  $\delta_1$  and  $\delta_2$  of modulus larger than 1, while

$$\text{spec}(D\mathcal{R}_{EKW}|_{\mathcal{B}_r(s_a)}) \setminus \{\delta_1, \delta_2\} \subset \{z \in \mathbb{C} : |z| \leq \nu\},$$

where

$$(15) \quad \nu < 0.85.$$

Finally, there is an  $s^* \in \mathcal{B}_r(s_a)$  such that

$$\mathcal{R}_{EKW}[s^*] = s^*.$$

The scalings  $\lambda^*$  and  $\mu^*$  corresponding to the fixed point  $s^*$  satisfy

$$(16) \quad \lambda_* \in [-0.24887681, -0.24887376],$$

$$(17) \quad \mu_* \in [0.061107811, 0.061112465].$$

**Remark 2.3.** The bound (15) is not sharp. In fact, a (lower) bound on the largest eigenvalue of  $D\mathcal{R}_{EKW}(s_*)$ , restricted to the tangent space of the stable manifold, is not known.

The interval enclosures of  $\lambda_*$  and  $\mu_*$  will be denoted

$$(18) \quad \boldsymbol{\lambda}_* \equiv [\lambda_-, \lambda_+]; \quad \lambda_- = -0.24887681, \quad \lambda_+ = -0.24887376,$$

$$(19) \quad \boldsymbol{\mu}_* \equiv [\mu_-, \mu_+]; \quad \mu_- = 0.061107811, \quad \mu_+ = 0.061112465.$$

The corresponding interval enclosure for the linear map

$$\Lambda_* \equiv \begin{bmatrix} \lambda_* & 0 \\ 0 & \mu_* \end{bmatrix}$$

will be denoted  $\boldsymbol{\Lambda}_*$ ; if  $(x, u) \in \mathbb{C}^2$ , then

$$(20) \quad \boldsymbol{\Lambda}_*(x, u) \equiv \{(\lambda x, \mu u) \in \mathbb{C}^2 : \lambda \in \boldsymbol{\lambda}_*, \mu \in \boldsymbol{\mu}_*\}.$$

The bound on the fixed point generating function  $s^*$  will be called  $\mathbf{s}^*$ :

$$(21) \quad \mathbf{s}^* \equiv \{s \in \mathcal{A}(\rho) : \|s - s_a\|_{\rho} \leq r = 6.0 \times 10^{-7}\},$$

while the bound on the renormalization fixed point  $F_*$  will be referred to as  $\mathbf{F}_*$ :

$$(22) \quad \mathbf{F}_* \equiv \{F : (x, -s(y, x)) \mapsto (y, s(x, y)) : s \in \mathbf{s}^*\},$$

the third iterate of this bound will be referred to as  $\mathbf{G}_*$ .

It follows from Theorem 1, that there exists a codimension 2 local stable manifold  $W_{\text{loc}}^s(s^*) \subset \mathcal{B}_r(s_a)$ .

**Definition 2.4.** A reversible map  $F$  of the form (4) such that  $s \in W_{\text{loc}}^s(s^*)$  is called infinitely renormalizable. The set of all reversible infinitely renormalizable maps is denoted by  $\mathbf{W}$ .

**Definition 2.5.** The set of reversible maps  $F$  of the form (4) with  $s \in \mathcal{B}_\varrho(s^*)$  will be referred to as  $\mathbf{F}_*(\varrho)$ . Denote,

$$\mathbf{W}(\varrho) \equiv \mathbf{W} \cap \mathbf{F}_*(\varrho).$$

Naturally,  $\mathbf{W}(\varrho)$  is invariant under renormalization if  $\varrho$  is sufficiently small.

Compactness of  $D\mathcal{R}_{EKW}|_{\mathcal{B}_r(s_a)}$  implies that for any  $\omega \leq \nu$  there exists a submanifold  $\mathbf{W}_\omega \subset \mathbf{W}$  (of finite codimension in  $\mathbf{W}$ ) such that  $\|R^k[F] - F_*\|_\rho \leq \text{const } \omega^k$  for all  $F \in \mathbf{W}_\omega$ .

**Definition 2.6.** Define

$$\mathbf{W}_\omega(\varrho) = \mathbf{W}(\varrho) \cap \mathbf{W}_\omega.$$

### 3. HYPERBOLIC SETS FOR MAPS IN $\mathbf{F}_*$

In this Section we will recall some of our results from the satellite paper (Gaidashev and Johnson 2009).

We will start by introducing several classical definitions which will be helpful in understanding our Theorem 2 below.

**Definition 3.1.** Let  $\mathcal{M}$  be a smooth manifold, and let  $F$  be a diffeomorphism of an open subset  $\mathcal{U} \subset \mathcal{M}$  onto its image.

An invariant set  $\mathcal{C}$  is called hyperbolic for the map  $F$  if there is a Riemannian metric on a neighborhood  $\mathcal{U}$  of  $\mathcal{C}$ , and  $\beta < 1 < \delta$ , such that for any  $p \in \mathcal{C}$  and  $n \in \mathbb{N}$  the tangent space  $T_{F^n(p)}\mathcal{U}$  admits a decomposition in two equivariant subspaces:

$$T_{F^n(p)}\mathcal{M} = E_n^+ \oplus E_n^-, \quad DF(F^n(p))E_n^\pm = E_{n+1}^\pm,$$

on which the sequence of differentials is hyperbolic:

$$\|DF(F^n(p))|_{E_n^-}\| < \beta, \quad \|DF^{-1}(F^n(p))|_{E_{n+1}^+}\| < \delta^{-1}.$$

The hyperbolic set  $\mathcal{C}$  is called locally maximal, if there is a neighborhood  $\mathcal{V}$  of  $\mathcal{C}$  such that  $\mathcal{C} = \bigcap_{n \in \mathbb{Z}} F^n(\bar{\mathcal{V}})$ .

**Definition 3.2.** Let  $\{0, 1, \dots, N-1\}^\mathbb{Z}$  be the space of all two-sided sequences of  $N$  symbols:

$$\{0, 1, \dots, N-1\}^\mathbb{Z} = \{\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots) : \omega_i \in \{0, 1, \dots, N-1\}, i \in \mathbb{Z}\},$$

Define the Bernoulli shift on  $\{0, 1, \dots, N-1\}^\mathbb{Z}$  as

$$\sigma_N(\omega) = \omega', \quad \omega'_n = \omega_{n+1}.$$

**Definition 3.3.** Let  $\mathcal{X}$  be a metric space,  $\mathcal{A} \subset \mathcal{X}$  and  $d \in [0, \infty)$ . Suppose that  $\mathcal{B} = \{B_i\}$  is some cover of  $\mathcal{A}$  whose elements are open sets. We will denote

$$(23) \quad C_d[\mathcal{B}] \equiv \sum_i \text{diam}(B_i)^d.$$

The  $d$ -dimensional Hausdorff content of  $\mathcal{A}$  is defined as

$$(24) \quad C_d^H[\mathcal{A}] = \inf \{C_d[\mathcal{B}] : \mathcal{B} \text{ is a cover of } \mathcal{A}\}.$$

The Hausdorff dimension of  $\mathcal{A}$  is defined as

$$(25) \quad \dim_H(\mathcal{A}) = \inf \{d \geq 0 : C_d^H[\mathcal{A}] = 0\}.$$

In (Gaidashev and Johnson 2009) we have demonstrated that all maps in a neighborhood of the fixed point admit a hyperbolic set in their domain of analyticity.

**Theorem 2.** The following holds for all  $F \in \mathbf{F}_*$ .

i) There exist connected open sets  $\mathcal{D} \subset \mathbb{C}^2$  and  $\mathcal{D}_3 \subset \mathbb{C}^2$  such that the maps  $F$  and  $G \equiv F \circ F \circ F$  are analytic on  $\mathcal{D}$  and  $\mathcal{D}_3$ , respectively.

ii) The map  $F$  possesses a hyperbolic fixed point  $p_0 = p_0(F) \in \mathcal{D}$ , such that

- 1)  $\mathcal{P}_x p_0 \in (0.57761843, 0.57761989)$ , and  $\mathcal{P}_u p_0 = 0$ , where  $\mathcal{P}_{x,u}$  are projections on the  $x$  and  $u$  coordinates;
- 2)  $DF(p_0)$  has two negative eigenvalues.

$$\begin{aligned} e_+ &\in (-2.05763559, -2.05759928), \\ e_- &\in (-0.48601715, -0.48598084). \end{aligned}$$

iii) The map  $G$  admits a locally maximal invariant hyperbolic set  $\mathcal{C}_G$ :

$$\mathcal{C}_G = \bigcap_{n \in \mathbb{Z}} G^{-n}(\Delta),$$

and

$$G|_{\mathcal{C}_G} \underset{\text{homeo}}{\approx} \sigma_2|_{\{0,1\}^{\mathbb{Z}}},$$

where  $\Delta = \Delta_0 \cup \Delta_1$  and  $\Delta_0 \subset \mathcal{D}_3$ ,  $\Delta_1 \subset \mathcal{D}_3$  are compact sets, diffeomorphic to rectangles, with non-empty interior, that constitute a Markov partition for  $G|_{\mathcal{C}_G}$ .

Furthermore, the Hausdorff dimension of  $\mathcal{C}_G$  satisfies:

$$0.76594 > \dim_H(\mathcal{C}_G) > \varepsilon,$$

where  $\varepsilon \approx 0.00013 e^{-7499}$  is strictly positive.

iv) The local stable manifold  $\mathcal{W}_{\text{loc}}^s(p_0) \cap \Delta_0$  is a graph over the  $x$ -axis with the angle of the slope bounded away from 0 and  $\pi/2$ .

**Remark 3.4.** The bounds on the rectangles  $\Delta_0$  and  $\Delta_1$  of the Markov partition for  $\mathcal{C}_G$  are given in Table 1.

One can construct a convergent sequence of approximations of the hyperbolic sets  $\mathcal{C}_G$  in a straightforward way. Define recursively:

$$(26) \quad \mathcal{U}_G^1 \equiv G(\Delta) \cap G^{-1}(\Delta) \quad \text{and} \quad \mathcal{U}_G^k \equiv G(\mathcal{U}_G^{k-1}) \cap G^{-1}(\mathcal{U}_G^{k-1}).$$

Each of the sets  $\mathcal{U}_G^k$  contains  $2 \cdot 4^k$  components  $\mathcal{U}_G^{k,n}$ ,  $n = 1..2 \cdot 4^k$ . The following Lemma has been proved in (Gaidashev and Johnson 2009).

Component	Centre	“Stable” Scale	“Unstable” Scale
$\Delta'_0$	(0.670198, 0.0)	0.083	0.083
$\Delta'_1$	(-0.441811, 0.0)	0.0655	0.0655

TABLE 1. The rectangles that approximate the Markov partition for the horseshoe of  $G$ . The rectangles are spanned by vectors  $\mathbf{e}_0^s = (0.788578889012330, -0.614933602760558)$ ,  $\mathbf{e}_0^u = T(\mathbf{e}_0^s)$  and  $\mathbf{e}_1^s = (0.750925931392967773, 0.660386436536671957)$ ,  $\mathbf{e}_1^u = T(\mathbf{e}_1^s)$ , respectively. The length of the sides of the rectangles  $\Delta'_0$  and  $\Delta'_1$  is  $2 \cdot \text{stable/unstable scale} \cdot |\mathbf{e}_{0,1}^{u,s}|$ .

**Lemma 3.** *Let*

$$\rho_{k,n} = \sup_{B_\rho \subset \mathcal{U}_G^{k,n}}(\rho),$$

and set  $\rho_k = \min_n \{\rho_{k,n}\}$ . *There exist constants  $C > 0$  and  $c > 0$  such that*

$$\text{diam}(\mathcal{U}_G^{k,n}) \leq C\kappa_+^k, \quad \rho_k \geq c\kappa_-^k,$$

where  $\kappa_- = 0.0371$  and  $\kappa_+ = 0.1642$ .

In this paper we will complement Theorem 2, and show that the third iterate of  $F$  also supports a stable set in its domain of analyticity.

#### 4. STATEMENT OF RESULTS

Recall, that a map  $\mathcal{H} : \mathcal{X} \rightarrow \mathcal{Y}$  between two metric spaces  $\mathcal{X}$  and  $\mathcal{Y}$  is called *bi-Lipschitz*, if there is a constant  $C \geq 1$ , such that for any two points  $p$  and  $q$  in  $\mathcal{X}$

$$(27) \quad \frac{1}{C} \text{dist}_{\mathcal{X}}(p, q) \leq \text{dist}_{\mathcal{Y}}(\mathcal{H}(p), \mathcal{H}(q)) \leq C \text{dist}_{\mathcal{X}}(p, q),$$

i.e. if  $\text{dist}_{\mathcal{X}}(p, q)$  and  $\text{dist}_{\mathcal{Y}}(\mathcal{H}(p), \mathcal{H}(q))$  are *commensurate*. We denote commensurability of two quantities by “ $\asymp$ ”.

A classical result from analysis states that such maps preserve the Hausdorff dimension.

Consider the dyadic group,

$$(28) \quad \{0, 1\}^\infty = \varprojlim \{0, 1\}^n,$$

where  $\varprojlim$  stands for the inverse limit. An element  $w$  of the dyadic group can be represented as a formal power series  $w \rightarrow \sum_{k=0}^\infty w_{k+1}2^k$ . The *odometer*, or the *adding machine*,  $p : \{0, 1\}^\infty \rightarrow \{0, 1\}^\infty$  is the operation of adding 1 in this group.

We are now ready to state our main theorem.

**Main Theorem 1.** *There exists  $\varrho > 0$  such that any  $F \in \mathbf{W}(\varrho)$  admits a “stable” Cantor set  $\mathcal{C}_F^\infty \subset \mathcal{D}$ , that is the set on which the maximal Lyapunov exponent is equal to zero, with the following properties.*

- 1) *The set  $\mathcal{C}_F^\infty$  is a Hausdorff limit of invariant hyperbolic sets of vanishing hyperbolicity.*
- 2) *The Hausdorff dimension of  $\mathcal{C}_F^\infty$  satisfies*

$$\dim_H(\mathcal{C}_F^\infty) \leq 0.5324.$$

- 3) The restriction of the dynamics  $F|_{\mathcal{C}_F^\infty}$  is topologically conjugate to the adding machine.
- 4) If

$$\omega \leq \min \left\{ \frac{\mu_*}{|\lambda_*|}, \frac{\inf_{p \in \mathcal{D}, v \in \mathbb{R}^2} \|\Lambda_* \cdot DF_*(p) \cdot v\|}{\sup_{p \in \mathcal{D}, v \in \mathbb{R}^2} \|\Lambda_* \cdot DF_*(p) \cdot v\|} \right\},$$

then for all  $F$  and  $\tilde{F}$  in  $\mathbf{W}_\omega(\varrho)$

$$\dim_H(\mathcal{C}_F^\infty) = \dim_H(\mathcal{C}_{\tilde{F}}^\infty),$$

and

$$F|_{\mathcal{C}_F^\infty} \approx_{\mathcal{H}} \tilde{F}|_{\mathcal{C}_{\tilde{F}}^\infty},$$

where  $\mathcal{H}$  is a bi-Lipschitz map.

**Remark 4.1.** We have obtained the following rigorous computer bound

$$\frac{\inf_{p \in \mathcal{D}, v \in \mathbb{R}^2} \|\Lambda_* \cdot DF_*(p) \cdot v\|}{\sup_{p \in \mathcal{D}, v \in \mathbb{R}^2} \|\Lambda_* \cdot DF_*(p) \cdot v\|} \leq 0.0581,$$

which is clearly dependent on our choice of coordinates for  $F$ 's. At the same time,  $\omega \leq \mu_*/|\lambda_*| < 0.246$  (independently of the choice of a coordinate system in which maps  $F$  are considered), while  $\nu$ , the renormalization convergence rate on  $\mathbf{W}(\varrho)$ , is less than 0.85. Therefore, it seems likely that the submanifold  $\mathbf{W}_\omega(\varrho) \neq \mathbf{W}(\varrho)$ .

## 5. SOME NOTATION AND DEFINITIONS

We will use the following notation for the sup norm of a function  $h$  and a transformation  $H$  defined on some set  $\mathcal{S} \subset \mathbb{R}^2$  or  $\mathbb{C}^2$ :

$$(29) \quad |h|_{\mathcal{S}} \equiv \sup_{(x,u) \in \mathcal{S}} \{|h|\},$$

$$(30) \quad |H|_{\mathcal{S}} \equiv \max \left\{ \sup_{(x,u) \in \mathcal{S}} \{|\mathcal{P}_x H|\}, \sup_{(x,u) \in \mathcal{S}} \{|\mathcal{P}_u H|\} \right\},$$

where  $\mathcal{P}_x$  and  $\mathcal{P}_u$  are projections on the corresponding components. We will also use the notation  $|\cdot|$  for the  $l_2$  norm for vectors in  $\mathbb{R}^2$ .

With  $D\mathbf{G}_* : p \mapsto D\mathbf{G}_*(p)$  we denote an interval matrix valued function such that

$$[DG(p)]_{ij} \in [D\mathbf{G}_*(p)]_{ij}, \quad \text{for all } G \in \mathbf{G}_*, p \in \mathcal{D}_3,$$

where  $\mathcal{D}_3$  is the domain of  $\mathbf{G}_*$ , and the bound on the operator norm of  $DG$  for  $G = F \circ F \circ F$ ,  $F \in \mathbf{F}_*$  on a set  $\mathcal{S}$  will be denoted

$$\|D\mathbf{G}_*\|_{\mathcal{S}} \equiv \sup_{F \in \mathbf{F}_*} \{\|D(F \circ F \circ F)\|_{\mathcal{S}}\}.$$

We will also use the following abbreviations for maps, transformations and scalings

$$(31) \quad G_j \equiv R^j[G],$$

$$(32) \quad \Lambda_{k,G} \equiv \Lambda_G \circ \Lambda_{R[G]} \circ \dots \circ \Lambda_{R^{k-1}[G]} = \Lambda_{G_0} \circ \Lambda_{G_1} \circ \dots \circ \Lambda_{G_{k-1}},$$

$$(33) \quad \lambda_{k,G} \equiv \lambda_{G_0} \lambda_{G_1} \dots \lambda_{G_{k-1}},$$

$$(34) \quad \mu_{k,G} \equiv \mu_{G_0} \mu_{G_1} \dots \mu_{G_{k-1}}.$$

## 6. A STABLE INVARIANT SET AS A HAUSDORFF LIMIT OF HYPERBOLIC SETS

According to Theorem 2 the fixed point  $F_*$  possesses a hyperbolic set for its third iterate. By the stability property of such sets, there exists a neighborhood  $\mathcal{B}_{r'}(s^*)$  such that all maps  $F$  of the form (4) with  $s \in \mathcal{B}_{r'}(s^*)$  also have a hyperbolic set  $\mathcal{C}_G$  for  $G \equiv F \circ F \circ F$ , and the action of  $G$  on  $\mathcal{C}_G$  is topologically conjugate to that of  $G_* \equiv F_* \circ F_* \circ F_*$  on  $\mathcal{C}_*$ :

$$G \circ H_G = H_G \circ G_*|_{\mathcal{C}_*}.$$

In what follows, we consider maps  $F \in \mathbf{W}(\varrho)$  (see Definition 2.5) where

$$\varrho \leq \min\{r', r\}.$$

The following holds on  $\Lambda_*^k(\mathcal{C}_*)$  for all  $F \in \mathbf{W}(\varrho)$ :

$$\begin{aligned} \Lambda_{k,G} \circ H_{G_k} \circ \Lambda_*^{-k} &= \Lambda_{k,G} \circ G_k^{-1} \circ H_{G_k} \circ G_* \circ \Lambda_*^{-k} \\ &= \Lambda_{k,G} \circ G_k^{-1} \circ \Lambda_{k,G}^{-1} \circ \Lambda_{k,G} \circ H_{G_k} \circ \Lambda_*^{-k} \circ \Lambda_*^k \circ G_* \circ \Lambda_*^{-k} \\ &= G^{-2^k} \circ \Lambda_{k,G} \circ H_{G_k} \circ \Lambda_*^{-k} \circ G_*^{2^k}. \end{aligned}$$

Therefore, the transformation

$$H_{k,G} = \Lambda_{k,G} \circ H_{G_k} \circ \Lambda_*^{-k}$$

is a topological conjugacy of iterates  $G^{2^k}$  and  $G_*^{2^k}$  on  $\Lambda_*^k(\mathcal{C}_*)$ :

Define

$${}_i H_{k,G} \equiv G^i \circ H_{k,G} \circ G_*^{-i}, \quad 0 \leq i \leq 2^k - 1.$$

Clearly,

$$(35) \quad G^{2^k} \circ {}_i H_{k,G} = {}_i H_{k,G} \circ G_*^{2^k} \quad \text{and} \quad G \circ {}_i H_{k,G} = {}_{i+1} H_{k,G} \circ G_*.$$

on  $\mathcal{C}_*^{k,i} \equiv \mathcal{C}_G^{k,i}$ , where

$$\begin{aligned} \mathcal{C}_G^{k,i} &\equiv G^i(H_{k,G}(\Lambda_*^k(\mathcal{C}_*))) \\ &= G^i(\Lambda_{k,G}(H_{G_k}(\mathcal{C}_*))) \\ &= G^i(\Lambda_{k,G}(\mathcal{C}_{G_k})) \\ (36) \quad &= {}_i H_{k,G}(G_*^i(\Lambda_*^k(\mathcal{C}_*))). \end{aligned}$$

Also, define the following sequence of sets

$$(37) \quad \mathcal{C}_G^k \equiv \bigcup_{i=0}^{2^k-1} \mathcal{C}_G^{k,i},$$

and their covers  $\mathcal{V}_G^k \supset \mathcal{C}_G^k$ :

$$\mathcal{V}_G^k \equiv \bigcup_{i=0}^{2^k-1} \mathcal{V}_G^{k,i},$$

where

$$\mathcal{V}_G^{k,0} \equiv \Lambda_{k,G}(\mathcal{U}_{G_k}^k),$$

$\mathcal{U}_G^k$  is as in (26), and

$$\mathcal{V}_G^{k,i} \equiv G^i(\Lambda_{k,G}(\mathcal{U}_{G_k}^k)) \cap G^{i-2^k}(\Lambda_{k,G}(\mathcal{U}_{G_k}^k)).$$

Clearly, the map  $\mathcal{H}_{k,G}$  defined as

$$\mathcal{H}_{k,G}|_{\mathcal{C}_*^{k,i}} \equiv {}_i H_{k,G},$$

is a conjugacy of  $G$  and  $G_*$  on  $\mathcal{C}_*^k$ :

$$(38) \quad G \circ \mathcal{H}_{k,G}|_{\mathcal{C}_*^k} = \mathcal{H}_{k,G} \circ G_*|_{\mathcal{C}_*^k}.$$

In the rest of this Section we will be studying the sequence of sets  $\mathcal{C}_G^k$ . We will demonstrate that the limit set  $\mathcal{C}_G^\infty$  exists, is *stable*, in the sense that the maximal Lyapunov exponent on  $\mathcal{C}_G^\infty$  is zero, bounded, closed and *invariant* under  $G$ . We will also show that there exists an  $\omega > 0$  such that for any  $F \in \mathbf{W}_\omega(\varrho)$  the sets  $\mathcal{C}_G^\infty$  are *weakly rigid*: there exists a bi-Lipschitz (see (27)) conjugacy  $\mathcal{H}_G$  between  $\mathcal{C}_*^\infty$  and  $\mathcal{C}_G^\infty$ .

The stable set  $\mathcal{C}_G^\infty$  is an analogue of the attracting set for dissipative Hénon-like maps constructed (de Carvalho *et al* 2005). The (more standard) approach of (de Carvalho *et al* 2005) is based on the so called presentation functions; it also demonstrates that the attracting set is Cantor and that the restriction of the dynamics to it is homeomorphic to the adding machine. We outline a similar procedure in Section 7.

The approach of this Section is more technical than that method of presentation functions; one of our goals in pursuing it was to demonstrate that *the stable set is a Hausdorff limit of hyperbolic Cantor sets with vanishing hyperbolicity*. On the other hand the more compact method of presentation functions shows that *the stable set is indeed Cantor, and that the stable dynamics is that of an odometer (an adding machine)*.

We will first demonstrate boundedness of sets  $\mathcal{C}_G^k$ .

Given a set  $\mathcal{S} \subset \mathcal{D}_3$  on which an iterate  $G^i$  for all  $G \in \mathbf{G}_*$  is defined, we use the notation  $\mathbf{G}_*^i(\mathcal{S})$  as a shorthand for  $\cup_{G \in \mathbf{G}_*} G^i(\mathcal{S})$ .

Notations  $\mathbf{\Lambda}_*^n(\mathcal{S})$  and  $\mathbf{T}_n(\mathcal{S})$  are used in a similar sense.

**Lemma 6.1.** *For all  $F \in \mathbf{W}(\varrho)$ , the sets  $\mathcal{C}_G^k$  are bounded, in particular,  $\mathcal{C}_G^k \subset \mathcal{E}$  for all  $k \geq 1$ , where*

$$\mathcal{E} \equiv \bigcup_{i=1}^4 \mathcal{E}_i,$$

and

$$\begin{aligned} \mathcal{E}_1 &= \{(x, u) \in \mathbb{R}^2 : \frac{(x + 0.0328)^2}{0.169^2} + \frac{u^2}{0.010683153^2} < 1\}, \\ \mathcal{E}_2 &\equiv \mathbf{G}_*(\mathcal{E}_1), \quad \mathcal{E}_3 \equiv \mathbf{\Lambda}_*(\mathcal{E}_2 \cup \mathcal{E}_4), \quad \mathcal{E}_4 \equiv T(\mathbf{G}_*(\mathcal{E}_1)). \end{aligned}$$

*Proof.* Let  $i < 2^k$  for some  $k \in \mathbb{N}$ . We write  $i$  in its binary representation:

$$i = \alpha_0 2^0 + \alpha_1 2^1 + \dots + \alpha_{k-1} 2^{k-1}, \quad \alpha_j = 0, 1.$$

Let  $\{j_l\}_{l=1}^m$ , be the index set such that  $\alpha_{j_l} \neq 0$ :

$$(39) \quad i = 2^{j_1} + 2^{j_2} + \dots + 2^{j_m}, \quad j_m \leq k-1, \quad m \leq k.$$

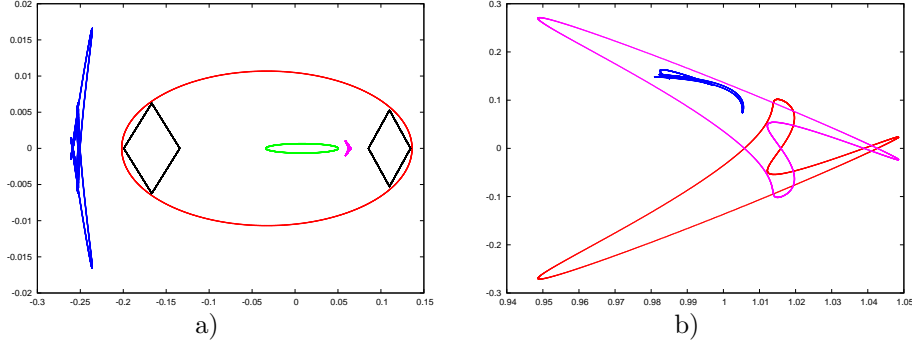


FIGURE 2. a) Sets  $\mathcal{E}_1$  (red),  $\mathcal{E}_3$  (blue),  $\Lambda_*(\mathcal{E}_1)$  (green) and  $\Lambda_*(\mathcal{E}_3)$  (magenta). The two components of the Markov partition, rescaled by  $\Lambda_*$ , are drawn in black. b) Sets  $\mathcal{E}_2$  (red),  $\mathcal{E}_4$  (magenta) and  $G_*(\mathcal{E}_3)$  (blue).

Consider  $G^i$  on a subset of  $\mathcal{D}_3$  where this iterate is defined:

$$\begin{aligned}
 G^i &= G^{2^{j_1}} \circ G^{2^{j_2}} \circ \dots \circ G^{2^{j_m}} \\
 &= \Lambda_{j_1, G} \circ \left[ \Lambda_{j_1, G}^{-1} \circ G^{2^{j_1}} \circ \Lambda_{j_1, G} \right] \circ \Lambda_{j_1, G}^{-1} \circ \Lambda_{j_2, G} \\
 &\quad \circ \left[ \Lambda_{j_2, G}^{-1} \circ G^{2^{j_2}} \circ \Lambda_{j_2, G} \right] \circ \Lambda_{j_2, G}^{-1} \circ \dots \circ \Lambda_{j_m, G} \\
 &\quad \circ \left[ \Lambda_{j_m, G}^{-1} \circ G^{2^{j_m}} \circ \Lambda_{j_m, G} \right] \circ \Lambda_{j_m, G}^{-1} \\
 &= \Lambda_{j_1, G} \circ G_{j_1} \circ \Lambda_{j_1, G}^{-1} \circ \Lambda_{j_2, G} \circ G_{j_2} \circ \dots \circ \Lambda_{j_{m-1}, G}^{-1} \circ \Lambda_{j_m, G} \circ G_{j_m} \circ \Lambda_{j_m, G}^{-1}.
 \end{aligned}$$

For convenience, we will denote

$$T_{n, m, G} = \Lambda_{m, G}^{-1} \circ \Lambda_{n, G} \circ G_n, \quad T_{n, 0, G} = \Lambda_{n, G} \circ G_n, \quad T_n = \Lambda_*^n \circ G_*, \quad \mathbf{T}_n \equiv \Lambda_*^n \circ \mathbf{G}_*,$$

and also use the following notation for compositions of these maps:

$$(40) \quad T_{[i]_q, G} = T_{j_q, j_{q-1}, G} \circ T_{j_{q+1}, j_q, G} \circ \dots \circ T_{j_l, j_{l-1}, G},$$

$$(41) \quad T_{[i]_q}^l = T_{j_q - j_{q-1}} \circ T_{j_{q+1} - j_q} \circ \dots \circ T_{j_l - j_{l-1}},$$

$$(42) \quad \mathbf{T}_{[i]_q}^l = \mathbf{T}_{j_q - j_{q-1}} \circ \mathbf{T}_{j_{q+1} - j_q} \circ \dots \circ \mathbf{T}_{j_l - j_{l-1}},$$

$$(43) \quad T_{[i], G} = T_{[i]_1^m, G}, \quad T_{[i]} = T_{[i]_1^m}, \quad \mathbf{T}_{[i]} = \mathbf{T}_{[i]_1^m},$$

where  $j_0 \equiv 0$ . In this notation, the iterate  $G^i$  can be written as

$$(44) \quad G^i = T_{[i], G} \circ \Lambda_{j_m, G}^{-1}.$$

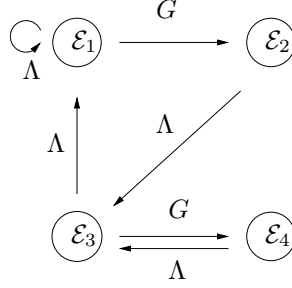
We apply the formula (44) to write the action of  $G^i$  on  $H_{k, G}(\Lambda_*^k(\mathcal{C}_*))$ :

$$G^i(H_{k, G}(\Lambda_*^k(\mathcal{C}_*))) = T_{[i], G} \circ \Lambda_{j_m, G}^{-1}(\Lambda_{k, G}(H_{G_k}(\mathcal{C}_*))) \subset \mathbf{T}_{[i]} \circ \Lambda_*^{k-j_m}(H_{G_k}(\mathcal{C}_*)).$$

The set  $\mathcal{E}_1$  was chosen so that  $\Lambda_*(\Delta) \subset \mathcal{E}_1$ , where  $\Delta$  is as in Theorem 2, and  $\Lambda_*(\mathcal{E}_1) \subset \mathcal{E}_1$ . Therefore,

$$G^i(H_{k, G}(\Lambda_*^k(\mathcal{C}_*))) \subset \mathbf{T}_{[i]}(\mathcal{E}_1).$$

Now, to demonstrate the invariance of the set  $\mathcal{E}$ , we verify that

FIGURE 3. Invariance of the set  $\mathcal{E}$  under the action of  $\mathbf{T}_n$ .

$$\Lambda_*(\mathcal{E}_1) \subseteq \mathcal{E}_1, \quad \Lambda_*(\mathcal{E}_3) \subseteq \mathcal{E}_1, \quad \mathbf{G}_*(\mathcal{E}_3) \subseteq \mathcal{E}_4$$

(see (Progs 2009) for programs used in this verification).

These inclusions imply (see Figure 3) that for any sequence  $\{j_n\}_{q-1}^l$ ,  $0 \leq j_{q-1} < j_q < \dots < j_l \leq k-1$ , the set  $\mathbf{T}_{[i]_q^l}(\mathcal{E}_1)$  is compactly contained in  $\mathcal{E}$ . The set  $\mathcal{E}$  is depicted in Figure 2.

□

The next technical property of the set  $\mathcal{E}$  will be required in the proof of weak rigidity in Section 8. This “separation” property has been verified directly on the computer.

**Lemma 6.2.** *The projections of sets  $\mathcal{E}_2$  and  $\mathcal{E}_4$  on the horizontal axis are separated from 0 and from rescalings of themselves:  $\text{dist}(\mathcal{P}_x(\mathcal{E}_2 \cup \mathcal{E}_4), 0)$  is strictly positive, and*

$$|\lambda_-| \sup_{p \in \mathcal{E}_2 \cup \mathcal{E}_4} |\mathcal{P}_x p| < \inf_{p \in \mathcal{E}_2 \cup \mathcal{E}_4} |\mathcal{P}_x p|.$$

**Remark 6.3.** *We have computed (see (Progs 2009)) the upper and lower bounds on the norms of  $\|D\mathbf{T}_1 v\|$  and  $\|D(\mathbf{G}_* \circ \Lambda_*)\|$  to be as follows:*

$$(45) \quad \|D\mathbf{T}_1\|_{\mathcal{E}_1} < 0.764 \equiv A_1, \quad \|D\mathbf{T}_1\|_{\mathcal{E}_3} < 0.344 \equiv A_3,$$

$$(46) \quad \|D(\mathbf{G}_* \circ \Lambda_*)\|_{\mathcal{E}} < 0.585 \equiv a, \quad \inf_{v \in \mathbb{R}^*, \|v\|=1} \|D\mathbf{T}_1 v\|_{\mathcal{E}_1 \cup \mathcal{E}_3} > 0.034 \equiv b,$$

We will also denote

$$A = \max\{A_1, A_3\}.$$

The next Lemma, albeit straightforward, will be important in our proofs of convergence of sets  $\mathcal{C}_G^k$  and existence of a bi-Lipschitz conjugacy between the limit sets.

**Lemma 6.4.** *There exist  $\varrho > 0$  and a function  $C(\varrho)$  with the property*

$$\lim_{\varrho \rightarrow 0} C(\varrho) = 0,$$

*such that for any  $F \in \mathbf{W}(\varrho)$*

$$(47) \quad |\mathcal{P}_x(H_{k,G} - \Lambda_{k,G} \circ \Lambda_*^{-k})|_{\Lambda_*^k(\mathcal{C}_*)} \leq C(\varrho) |\lambda_{k,G}| \nu^k,$$

$$(48) \quad |\mathcal{P}_u(H_{k,G} - \Lambda_{k,G} \circ \Lambda_*^{-k})|_{\Lambda_*^k(\mathcal{C}_*)} \leq C(\varrho) \mu_{k,G} \nu^k.$$

Furthermore, if  $F \in \mathbf{W}_\omega(\varrho)$  then

$$(49) \quad |\mathcal{P}_x(H_{k,G} - \Lambda_{k,G} \circ \Lambda_*^{-k})|_{\Lambda_*^k(\mathcal{C}_*)} \leq C(\varrho) |\lambda_{k,G}| \omega^k,$$

$$(50) \quad |\mathcal{P}_u(H_{k,G} - \Lambda_{k,G} \circ \Lambda_*^{-k})|_{\Lambda_*^k(\mathcal{C}_*)} \leq C(\varrho) \mu_{k,G} \omega^k.$$

*Proof.* By the strong structural stability property of the hyperbolic sets (see e.g. Theorem 18.1.3 and 18.2.1 in (Katok and Hasselblatt 1995)),  $\|G_k - G_*\|_{\mathcal{D}_3} \xrightarrow{k \rightarrow \infty} 0$  implies

$$(51) \quad |H_{G_k} - Id|_{\mathcal{C}_*} \xrightarrow{k \rightarrow \infty} 0,$$

and, in fact, if  $\varrho$  is sufficiently small then there exists a constant  $C'$  such that for all  $F \in \mathbf{W}(\varrho)$

$$|H_{G_k} - Id|_{\mathcal{C}_*} < C' \|G_k - G_*\|_{\mathcal{D}_3}.$$

Now, for all  $F \in \mathbf{W}(\varrho)$

$$\|G_k - G_*\|_{\mathcal{D}_3} \leq C''(\varrho) \nu^k,$$

where the “constant”  $C''(\varrho)$  decreases to zero with the size of the local manifold  $\mathbf{W}(\varrho)$ , therefore,

$$(52) \quad |H_{G_k} - Id|_{\mathcal{C}_*} \leq C(\varrho) \nu^k$$

for some function  $C(\varrho)$  with the property  $\lim_{\varrho \rightarrow 0} C(\varrho) = 0$ . In a similar way, if  $F \in \mathbf{W}_\omega(\varrho)$ , then

$$(53) \quad |H_{G_k} - Id|_{\mathcal{C}_*} \leq C(\varrho) \omega^k.$$

Finally,

$$(54) \quad \begin{aligned} |\mathcal{P}_x(H_{G_k} - Id)|_{\mathcal{C}_*} &= |\mathcal{P}_x(\Lambda_{k,G}^{-1} \circ H_{k,G} \circ \Lambda_*^k - \Lambda_*^{-k} \circ \Lambda_*^k)|_{\mathcal{C}_*} \\ &= |\lambda_{k,G}|^{-1} |\mathcal{P}_x(H_{k,G} - \Lambda_{k,G} \circ \Lambda_*^{-k})|_{\Lambda_*^k(\mathcal{C}_*)}, \end{aligned}$$

and similarly for  $\mathcal{P}_u(H_{G_k} - Id)$ . The claim follows.  $\square$

In several following propositions and theorems we will have to use a number of “constants”  $c_i(\varrho)$  all of which have the property

$$\lim_{\varrho \rightarrow 0} c_i(\varrho) = 0.$$

**Proposition 6.5.** *There exists  $\varrho > 0$  such that for all  $F \in \mathbf{W}(\varrho)$  the sets  $\mathcal{V}_G^k$  and  $\mathcal{C}_G^k$  converge in the Hausdorff metric, specifically:*

$$d_H(\mathcal{V}_G^k, \mathcal{V}_G^{k+1}) \leq \text{const } \theta^k, \quad d_H(\mathcal{C}_G^k, \mathcal{C}_G^{k+1}) \leq \text{const } \theta^k,$$

where  $\theta = 0.436$ .

Furthermore, for any fixed  $i$ , there is  $K > 0$ , such that for all  $k \geq K$

$$(55) \quad d_H(\mathcal{V}_G^{k,i}, G^i((0,0))) \leq \text{const } \theta^k,$$

the limit set is closed, and satisfies

$$\mathcal{C}_G^\infty = \overline{\bigcup_{i \in \mathbb{Z}} G^i((0,0))}.$$

*Proof.* Clearly,

$$d_H(\mathcal{V}_G^{k,0}, \mathcal{V}_G^{k+1,0}) = d_H(\Lambda_{k,G}(\mathcal{U}_{G_k}^k), \Lambda_{k+1,G}(\mathcal{U}_{G_{k+1}}^{k+1})) \leq \text{const} |\lambda_-|^k.$$

Let the binary expansion of  $i < 2^k$  be as in (39). Recall, that according to Lemma 6.1

$$\begin{aligned} G^i(\Lambda_{k,G}(\mathcal{U}_{G_k}^k)) &\subset \mathbf{T}_{[i]} \circ \Lambda_*^{k-j_m}(\mathcal{U}_{G_k}^k), \\ G^i(\Lambda_{k+1,G}(\mathcal{U}_{G_{k+1}}^{k+1})) &\subset \mathbf{T}_{[i]} \circ \Lambda_*^{k+1-j_m}(\mathcal{U}_{G_{k+1}}^{k+1}), \end{aligned}$$

where  $\mathbf{T}_{[i]}$  is as in (43).

Let  $s_k \equiv \Lambda_{k,G}(s) \in \mathcal{V}_G^{k,0}$  and  $p_{k+1} \equiv \Lambda_{k+1,G}(p) \in \mathcal{V}_G^{k+1,0}$  be any two points in the corresponding sets. Since  $j_m \leq k-1$ , both  $\Lambda_*^{k-j_m}(s) \subset \Lambda_*^{k-j_m}(\mathcal{U}_{G_k}^k)$  and  $\Lambda_*^{k+1-j_m}(p) \subset \Lambda_*^{k+1-j_m}(\mathcal{U}_{G_{k+1}}^{k+1})$  are contained in  $\mathcal{E}_1$ . According to Lemma 6.1 the sequences

$$\mathbf{T}_{[i]_{m-l+1}^m} \circ \Lambda_*^{k-j_m}(s) \quad \text{and} \quad \mathbf{T}_{[i]_{m-l+1}^m} \circ \Lambda_*^{k+1-j_m}(p), \quad 1 \leq l \leq m,$$

land in  $\mathcal{E}_1$  if  $j_{m-l+1} - j_{m-l} > 1$ , and in  $\mathcal{E}_3$  if  $j_{m-l+1} - j_{m-l} = 1$ . Suppose, out of  $m$  differences  $j_n - j_{n-1}$ ,  $n = 1, \dots, m$ ,  $q$  are larger than 1 and  $m-q$  equal to 1. Then,

$$\begin{aligned} |G^i(s_k) - G^i(p_{k+1})| &\leq \left| \mathbf{T}_{[i]} \circ \Lambda_*^{k-j_m}(s) - \mathbf{T}_{[i]} \circ \Lambda_*^{k+1-j_m}(p) \right| \\ (56) \quad &\leq |\lambda_-|^{k-m} \|D\mathbf{T}_1\|_{\mathcal{E}_1}^q \|D\mathbf{T}_1\|_{\mathcal{E}_3}^{m-q} |s - \Lambda_*(p)|, \end{aligned}$$

where

$$\begin{aligned} |\mathbf{T}_{[i]} \circ \Lambda_*^{k-j_m}(s) - \mathbf{T}_{[i]} \circ \Lambda_*^{k+1-j_m}(p)| &\equiv \\ &\equiv \sup_{G \in \mathbf{G}_*, \Lambda \in \Lambda_*} |T_{[i],G} \circ \Lambda^{k-j_m}(s) - T_{[i],G} \circ \Lambda^{k+1-j_m}(p)|. \end{aligned}$$

The more often  $\|D\mathbf{G}_*\|$  in  $\|D\mathbf{T}_1\|$  has to be evaluated on  $\mathcal{E}_1$ , that is, the more often the bound  $A_1$  (see Remark 6.3) appears in the product in (56), the worse the resultant bound. Recall that  $m \leq k$  and  $j_m \leq k-1$ . Therefore, if  $m \leq [\frac{k}{2}]$ , then all differences  $j_n - j_{n-1}$  may be larger than 1 ( $q = m$ ), and

$$|G^i(s_k) - G^i(p_{k+1})| < |\lambda_-|^{k-m} A_1^m |s - \Lambda_*(p)| < \text{const} |\lambda_-|^{[\frac{k}{2}]} A_1^{[\frac{k}{2}]}.$$

If  $m > [\frac{k}{2}]$  then there are at most  $q = k - m$  differences  $j_n - j_{n-1}$  that are larger than 1:

$$\begin{aligned} |G^i(s_k) - G^i(p_{k+1})| &< |\lambda_-|^{k-m} A_1^{k-m} A_3^{m-(k-m)} |s - \Lambda_*(p)| \\ &= \left[ \frac{A_3^2}{A_1 |\lambda_-|} \right]^m \left[ \frac{|\lambda_-| A_1}{A_3} \right]^k |s - \Lambda_*(p)|, \end{aligned}$$

and since  $A_3^2/|\lambda_-| A_1 < 1$  we get in this case

$$|G^i(s_k) - G^i(p_{k+1})| < \text{const} \left[ \frac{A_3^2}{A_1 |\lambda_-|} \right]^{[\frac{k}{2}]} \left[ \frac{|\lambda_-| A_1}{A_3} \right]^k = \text{const} \left[ |\lambda_-|^{1/2} A_1^{1/2} \right]^k.$$

Since  $|\lambda_-|^{1/2} A_1^{1/2} < 0.436 < 1$ , we get

$$|G^i(s_k) - G^i(p_{k+1})| < \text{const} 0.436^k, \quad 1 \leq i \leq 2^k - 1.$$

Any point in  $\mathcal{V}_G^{k,i}$  can be represented as  $G^i(s_k)$  for some  $s_k \in \mathcal{V}_G^{k,0}$ , and any point in  $\mathcal{V}_G^{k+1,i}$  can be represented as  $G^i(p_{k+1})$  for some  $p_{k+1} \in \mathcal{V}_G^{k+1,0}$ , therefore

$$d_H(\mathcal{V}_G^{k,i}, \mathcal{V}_G^{k+1,i}) < \text{const } 0.436^k, \quad i \leq 2^k - 1.$$

A similar computation holds for inverse iterates  $G^{-1} = T \circ G \circ T$ :

$$d_H(\mathcal{V}_G^{k,2^k-n}, \mathcal{V}_G^{k+1,2^{k+1}-n}) < \text{const } 0.436^k, \quad 1 \leq n \leq 2^k - 1.$$

This demonstrates that the Hausdorff distance between components  $\mathcal{V}_G^{k,i}$  and components  $\mathcal{V}_G^{k+1,i}$ ,  $\mathcal{V}_G^{k+1,2^k+i}$  decreases with  $k$  at a geometric rate.

An identical argument for sets  $\mathcal{C}_G^k$  (rather than  $\mathcal{V}_G^k$ ) shows that these sets converge in the Hausdorff metric at the same rate  $\theta$ . We define the set  $\mathcal{C}_G^\infty$  as the set of all limit points of sequences  $\{p_k\}$ ,  $p_k \in \mathcal{C}_G^k$ . Such set is clearly closed.

Finally, to show (55), we again notice that if  $s_{k,i} \in \mathcal{V}_G^{k,i}$ , then there exists a point  $s_k \equiv \Lambda_{k,G}(s) \in \mathcal{V}_G^{k,0}$  such that  $s_{k,i} = G^i(s_k)$ . Therefore, if  $K \in \mathbb{Z}$  is such that  $2^K > i$  then for any  $k > K$

$$|s_{k,i} - G^i((0,0))| = |G^i(s_k) - G^i((0,0))| \leq |\lambda_-|^{k-m} \|D\mathbf{T}_1\|_{\mathcal{E}_1}^q \|D\mathbf{T}_1\|_{\mathcal{E}_3}^{m-q} |s - (0,0)| < \text{const } \theta^k.$$

□

We will now show that the set  $\mathcal{C}_G^\infty$  is invariant for  $G$ .

**Lemma 6.6.** *For any  $F \in \mathbf{W}(\varrho)$  the sets  $\mathcal{C}_G^k$  are invariant under  $G$ . The same is true about the set  $\mathcal{C}_G^\infty$ .*

*Proof.* This follows from a simple computation:

$$\begin{aligned} G^{2^k}(H_{k,G}(\Lambda_*^k(\mathcal{C}_*))) &= G^{2^k}(\Lambda_{k,G}(\Lambda_{k,G}^{-1}(H_{k,G}(\Lambda_*^k(\mathcal{C}_*)))) \\ &= G^{2^k}(\Lambda_{k,G}(H_{G_k}(\mathcal{C}_*))) \\ &= \Lambda_{k,G}(G_k(H_{G_k}(\mathcal{C}_*))) \\ &\subset \Lambda_{k,G}(H_{G_k}(\mathcal{C}_*)) \\ &= H_{k,G}(\Lambda_*^k(\mathcal{C}_*)). \end{aligned}$$

By Proposition 6.5, a point  $p_\infty \in \mathcal{C}_G^\infty$  is a limit point of some sequence  $\{p_k\}$ ,  $p_k \in \mathcal{C}_G^k$ . Because of the invariance of  $\mathcal{C}_G^k$  we have that  $G(p_k) \in \mathcal{C}_G^k$  for all  $k \in \mathbb{N}$ . Analyticity of the map  $G$  implies that  $\{G(p_k)\}$  converges in  $\mathcal{C}_G^\infty$ :

$$G(p_\infty) = G(\lim_{k \rightarrow \infty} p_k) = \lim_{k \rightarrow \infty} G(p_k) \in \mathcal{C}_G^\infty.$$

□

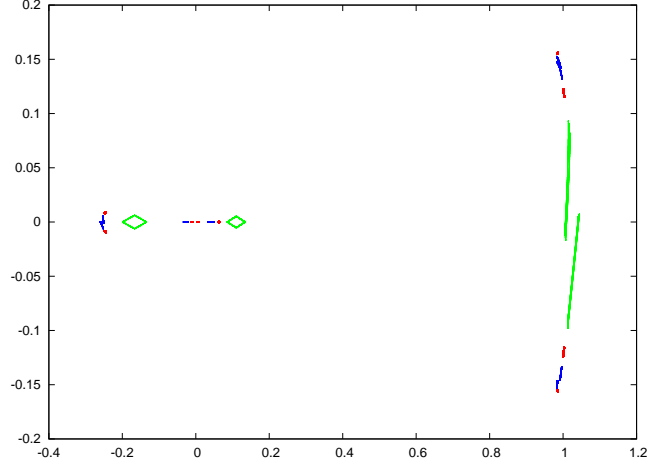
We will now address the convergence properties of transformations  ${}_i H_{k,G}$ .

**Proposition 6.7.** *There exists  $\varrho > 0$  such that for all  $F \in \mathbf{W}(\varrho)$  the following holds.*

1) *The transformations  $G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}$  are defined and analytic on  $\mathcal{V}_*^{k,i}$  for all  $k \in \mathbb{N}$  and  $0 \leq i < 2^k$ , and satisfy*

$$(57) \quad |{}_i H_{k,G} - G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}|_{\mathcal{C}_*^{k,i}} \leq C(\varrho) (\nu \theta)^k,$$

$$(58) \quad |G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i} - Id|_{\mathcal{V}_*^{k,i}} \leq c_1(\varrho),$$

FIGURE 4. Approximations of sets  $\mathcal{C}_G^1$  (green),  $\mathcal{C}_G^2$  (blue) and  $\mathcal{C}_G^3$  (red)

where  $C(\varrho)$  is as in Lemma 6.4, and  $c_1(\varrho)$  is some function of  $\varrho$  independent of  $k$ ,  $i$  and  $G$ , and satisfying

$$\lim_{\varrho \rightarrow 0} c_1(\varrho) = 0.$$

2) For any  $p \in \mathcal{V}_*^{k,i}$  and  $s \in \mathcal{V}_*^{k+1,i}$

$$(59) \quad |G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}(p) - G^i \circ \Lambda_{k+1,G} \circ \Lambda_*^{-k-1} \circ G_*^{-i}(s)| < \text{const } \theta^k.$$

*Proof.*

1) Notice that

$$G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}(G_*^i(\mathcal{V}_*^{k,0})) = G^i(\Lambda_{k,G}(\mathcal{U}_*^k)) \subset G^i(\Lambda_*^k(\mathcal{U}_*^k)).$$

By Lemma 6.1 the iterate  $G^i$ ,  $1 \leq i < 2^k$ , is well-defined and analytic on  $\Lambda_*^k(\mathcal{U}_*^k)$  for all  $G \in \mathbf{G}_*$ .

Proving (57) is similar to (56) and arguments that follow it:

$$\begin{aligned} & |{}_i H_{k,G} - G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}|_{G_*^i(\Lambda_*^k(\mathcal{C}_*))} \\ &= |G^i \circ H_{k,G} - G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k}|_{\Lambda_*^k(\mathcal{C}_*)} \\ &\leq \left| T_{[i],G} \circ \Lambda_{j_m,G}^{-1} \circ \Lambda_{k,G} \circ \Lambda_{k,G}^{-1} \circ H_{k,G} - T_{[i],G} \circ \Lambda_{j_m,G}^{-1} \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \right|_{\Lambda_*^k(\mathcal{C}_*)} \\ &\leq \theta^k \left| \Lambda_{k,G}^{-1} \circ H_{k,G} \circ \Lambda_*^k - Id \right|_{\mathcal{C}_*} \leq C(\varrho) (\nu \theta)^k, \end{aligned}$$

where the function  $C(\varrho)$  is as in Lemma 6.4.

We will now demonstrate (58) in two steps.

Step (1). Write

$$(60) \quad G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i} = \mathcal{J}_{G,m,i} \circ \left\{ G_{j_m} \circ \Lambda_{j_m,G}^{-1} \circ \Lambda_{k,G} \circ \Lambda_*^{j_m-k} \circ G_*^{-1} \right\} \circ \mathcal{J}_{G_*,m,i}^{-1},$$

where we have denoted for all  $1 \leq q \leq m$ :

$$\mathcal{J}_{G,q,i} \equiv T_{[i]_1^{q-1},G} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} = G^i \circ \Lambda_{j_m,G} \circ T_{[i]_{q+1}^m,G}^{-1} \circ G_{j_q}^{-1}.$$

The image of  $\mathcal{V}_*^{k,i}$  under the inverse of this map is contained in  $\mathcal{E}_2 \cup \mathcal{E}_4$  for all  $1 \leq q \leq m$ :

$$(61) \quad \mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i}) = G_* \circ T_{[i]_{q+1}^m} \circ \Lambda_*^{k-j_m}(\mathcal{U}_*^k) \subset \mathbf{G}_* \circ \mathbf{T}_{[i]_{q+1}^m} \circ \Lambda_*^{k-j_m}(\mathcal{U}_*^k) \Subset \mathcal{E}_2 \cup \mathcal{E}_4,$$

while

$$(62) \quad G_*^{-1}(\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})) \Subset \mathcal{E}_1 \cup \mathcal{E}_3.$$

Since  $|\lambda_{G_n} - \lambda_*| \leq c_2(\varrho) \nu^n$  and  $|\mu_{G_n} - \mu_*| \leq c_2(\varrho) \nu^n$ , we get

$$(63) \quad |\Lambda_{j_m,G}^{-1} \circ \Lambda_{k,G} \circ \Lambda_*^{j_m-k} - Id|_{G_*^{-1}(\mathcal{J}_{G_*,m,i}^{-1}(\mathcal{V}_*^{k,i}))} \leq c_3(\varrho) \nu^{j_m}.$$

Since containment of  $G_{j_q}^{-1}(\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i}))$  in  $\mathcal{E}_1 \cup \mathcal{E}_3$  is compact, it is possible to choose  $\varrho$  so that for all  $1 \leq q \leq m$

$$\Lambda_{j_q,G}^{-1} \circ \Lambda_{k,G} \circ \Lambda_*^{j_q-k}(G_*^{-1}(\mathcal{J}_{G_*,m,i}^{-1}(\mathcal{V}_*^{k,i}))) \subset \mathcal{E}_1 \cup \mathcal{E}_3.$$

The map  $G_{j_m}$  is defined and analytic on  $\mathcal{E}_1 \cup \mathcal{E}_3$  and maps it into  $\mathcal{E}_2 \cup \mathcal{E}_4$ , and therefore  $G_{j_m} \circ \Lambda_{j_m,G}^{-1} \circ \Lambda_{k,G} \circ \Lambda_*^{j_m-k} \circ G_*^{-1}$  is analytic on  $\mathcal{J}_{G_*,m,i}^{-1}(\mathcal{V}_*^{k,i})$  and maps it into  $\mathcal{E}_2 \cup \mathcal{E}_4$ . Because of (63) we also have for any  $n > j_q$ :

$$\begin{aligned} & |Id - G_{j_q} \circ \Lambda_{j_q,G}^{-1} \circ \Lambda_{n,G} \circ \Lambda_*^{j_q-n} \circ G_*^{-1}|_{\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})} \\ & \leq |G_{j_q} \circ \Lambda_{j_q,G}^{-1} \circ \Lambda_{n,G} \circ \Lambda_*^{j_q-n} \circ G_*^{-1} - G_{j_q} \circ G_*^{-1}|_{\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})} \\ & + |G_{j_q} \circ G_*^{-1} - Id|_{\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})} \\ (64) \quad & \leq \text{const } c_3(\varrho) \nu^{j_q} + c_4(\varrho) \nu^{j_q} \leq c_5(\varrho)(1 + \text{const}) \nu^{j_q} = c_6(\varrho) \nu^{j_q}, \end{aligned}$$

where  $c_4(\varrho)$  is another constant decreasing to zero together with  $\varrho$ , and  $c_5(\varrho)$  is the maximum of  $c_3(\varrho)$  and  $c_4(\varrho)$ .

As the result of the above discussion, we have

$$(65) \quad G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}|_{\mathcal{V}_*^{k,i}} = \mathcal{J}_{G,m,i} \circ \{Id + h_{j_m}\} \circ \mathcal{J}_{G_*,m,i}^{-1}|_{\mathcal{V}_*^{k,i}},$$

where  $h_{j_m}$  is some function analytic on  $\mathcal{J}_{G_*,m,i}^{-1}(\mathcal{V}_*^{k,i})$  and satisfying

$$(66) \quad |h_{j_m}|_{\mathcal{J}_{G_*,m,i}^{-1}(\mathcal{V}_*^{k,i})} \leq c_6(\varrho) \nu^{j_m}.$$

Step (2). At the next step, to obtain the bound (58) we will use an inductive argument.

Suppose that for  $q \leq m$

$$(67) \quad G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}|_{\mathcal{V}_*^{k,i}} = \mathcal{J}_{G,q,i} \circ \{Id + h_{j_q}\} \circ \mathcal{J}_{G_*,q,i}^{-1}|_{\mathcal{V}_*^{k,i}},$$

where  $h_{j_q}$  is some function analytic on  $\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})$  and satisfying

$$|h_{j_q}|_{\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})} \leq c_6(\varrho) \left[ \sum_{i=0}^{m-q} a^i \nu^{j_q+i-j_q} \right] \nu^{j_q}.$$

This is certainly satisfied for  $q = m$  (see (65) and (66)).

We prove that a representation similar to (67) holds for  $q-1$  with a similar bound on  $h_{j_{q-1}}$ . First,

$$\begin{aligned} (68) \quad G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}|_{\mathcal{V}_*^{k,i}} &= \mathcal{J}_{G,q-1,i} \circ \left\{ G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} \circ (Id + h_{j_q}) \right. \\ &\quad \left. \circ \Lambda_*^{j_{q-1}-j_q} \circ G_*^{-1} \right\} \circ \mathcal{J}_{G_*,q-1,i}^{-1}|_{\mathcal{V}_*^{k,i}}. \end{aligned}$$

Again, consider the map in the brackets:

$$\begin{aligned}
& |Id - G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} \circ (Id + h_{j_q}) \circ \Lambda_*^{j_{q-1}-j_q} \circ G_*^{-1}|_{\mathcal{J}_{G_*,m-1,i}^{-1}(\mathcal{V}_*^{k,i})} \\
& \leq |Id - G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} \circ \Lambda_*^{j_{q-1}-j_q} \circ G_*^{-1}|_{\mathcal{J}_{G_*,q-1,i}^{-1}(\mathcal{V}_*^{k,i})} \\
& + |G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} \circ (Id + h_{j_q}) \circ \Lambda_*^{j_{q-1}-j_q} \circ G_*^{-1} \\
(69) \quad & - G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} \circ \Lambda_*^{j_{q-1}-j_q} \circ G_*^{-1}|_{\mathcal{J}_{G_*,q-1,i}^{-1}(\mathcal{V}_*^{k,i})}.
\end{aligned}$$

The first norm in (69) has been estimated in (64). To provide a bound on the second norm we will use the fact that

$$G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} = \left\{ G_{j_{q-1}} \circ \Lambda_{G_{j_{q-1}}} \right\} \circ \left[ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q-1,G} \right],$$

and that if  $j_q - 1 - j_{q-1} = 0$  then

$$\Lambda_{j_{q-1},G}^{-1}(\Lambda_{j_q-1,G}(\mathcal{E}_2 \cup \mathcal{E}_4)) = \mathcal{E}_2 \cup \mathcal{E}_4,$$

if  $j_q - 1 - j_{q-1} = 1$  then

$$\Lambda_{j_{q-1},G}^{-1}(\Lambda_{j_q-1,G}(\mathcal{E}_2 \cup \mathcal{E}_4)) \subset \mathcal{E}_3,$$

if  $j_q - 1 - j_{q-1} \geq 2$  then

$$\Lambda_{j_{q-1},G}^{-1}(\Lambda_{j_q-1,G}(\mathcal{E}_2 \cup \mathcal{E}_4)) \subset \mathcal{E}_1$$

Therefore,

$$\begin{aligned}
& |Id - G_{j_{q-1}} \circ \Lambda_{j_{q-1},G}^{-1} \circ \Lambda_{j_q,G} \circ (Id + h_{j_q}) \circ \Lambda_*^{j_{q-1}-j_m} \circ G_*^{-1}|_{\mathcal{J}_{G_*,m-1,i}^{-1}(\mathcal{V}_*^{k,i})} \\
& \leq c_6(\varrho) \nu^{j_{q-1}} + |\lambda_-|^{j_q-1-j_{q-1}} \left\| D \left\{ G_{j_{q-1}} \circ \Lambda_{G_{j_{q-1}}} \right\} \right\|_{\mathcal{E}} |h_{j_q}|_{\mathcal{J}_{G_*,q,i}^{-1}(\mathcal{V}_*^{k,i})} \\
& \leq c_6(\varrho) \nu^{j_{q-1}} + ac_6(\varrho) \left[ \sum_{i=0}^{m-q} a^i \nu^{j_q+i-j_q} \right] \nu^{j_q} \\
& = c_6(\varrho) \left[ \sum_{i=0}^{m-q+1} a^i \nu^{j_{q-1}+i-j_{q-1}} \right] \nu^{j_{q-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |Id - G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}|_{\mathcal{V}_*^{k,i}} \\
& = |Id - \mathcal{J}_{G,1,i} \circ \{Id + h_{j_1}\} \circ \mathcal{J}_{G_*,1,i}^{-1}|_{\mathcal{V}_*^{k,i}} \\
& = |Id - \Lambda_{j_1,G} \circ \{Id + h_{j_1}\} \circ \Lambda_*^{-j_1}|_{\mathcal{V}_*^{k,i}} \\
& = |Id - \Lambda_{j_1,G} \circ \Lambda_*^{-j_1}|_{\mathcal{V}_*^{k,i}} + |\Lambda_{j_1,G} \circ h_{j_1}|_{\mathcal{J}_{G_*,1,i}^{-1}(\mathcal{V}_*^{k,i})} \\
& \leq c_3(\varrho) + |\lambda_-^{j_1}| c_6(\varrho) \left[ \sum_{i=0}^{m-1} a^i \nu^{j_1+i-j_1} \right] \nu^{j_1} \\
& \leq c_3(\varrho) + c_6(\varrho) |\lambda_- \nu|^{j_1} \left[ \sum_{i=0}^{k-1} (a\nu)^i \right] \\
& \leq c_3(\varrho) + c_6(\varrho) [1 - a\nu]^{-1} \equiv c_1(\varrho),
\end{aligned}$$

the last equality being the definition of  $c_1(\varrho)$ .

2) To demonstrate (59) we notice that  $G_*^{-i}(p)$  is in  $\mathcal{V}_*^{k,0}$  and  $G_*^{-i}(s)$  is in  $\mathcal{V}_*^{k+1,0}$ , while  $\hat{p} = \Lambda_*^{-k}(G_*^{-i}(p))$  and  $\hat{s} = \Lambda_*^{-k-1}(G_*^{-i}(s))$  are in  $\mathcal{U}_*^k$ . It follows from a computation similar to (32) that

$$\begin{aligned} & |G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}(p) - G^i \circ \Lambda_{k+1,G} \circ \Lambda_*^{-k-1} \circ G_*^{-i}(s)| = \\ & |G^i \circ \Lambda_{k,G}(\hat{p}) - G^i \circ \Lambda_{k+1,G}(\hat{s})| \\ & \leq \left| \mathbf{T}_{[i]} \circ \Lambda_*^{k-j_m}(\hat{p}) - \mathbf{T}_{[i]} \circ \Lambda_*^{k+1-j_m}(\hat{s}) \right| \\ & \leq \text{const } \theta^k. \end{aligned}$$

□

The above proposition implies, that if  $p$  is in the limit set  $\mathcal{C}_*^\infty$ , then there exist integers  $i$  and  $K$ , dependent on  $p$ , and a sequence of points  $p_{k,i} \in \mathcal{C}_*^{k,i}$ ,  $k \geq K$ , that converge to  $p$ :  $\lim_{k \rightarrow \infty} p_{k,i} = p$ . We have from (38)

$$G \circ \mathcal{H}_{k,G}(p_{k,i}) = \mathcal{H}_{k,G} \circ G_*(p_{k,i}).$$

Bounds (57)–(59) imply that the limit

$$(70) \quad \mathcal{H}_G(p) \equiv \lim_{k \rightarrow \infty} \mathcal{H}_{k,G}(p_{k,i}) = \lim_{k \rightarrow \infty} G^i \circ \Lambda_{k,G} \circ \Lambda_*^{-k} \circ G_*^{-i}(p_{k,i})$$

exists.

We will finally demonstrate that the limit set  $\mathcal{C}_G^\infty$  is stable.

Recall, the definition of the upper Lyapunov exponent of  $(p, v) \in (\mathcal{D} \cap \mathbb{R}^2) \times \mathbb{R}^2$  with respect to  $G$ :

$$\chi(p, v; G) \equiv \overline{\lim}_{i \rightarrow \infty} \frac{1}{i} \log [\|DG^i(p)v\|],$$

where  $\|\cdot\|$  is some norm in  $\mathbb{R}^2$ . The maximal Lyapunov exponent of  $p \in (\mathcal{D} \cap \mathbb{R}^2)$  with respect to  $G$  is defined as

$$\chi(p; G) \equiv \sup_{\|v\|=1} \chi(p, v; G).$$

**Lemma 6.8.** *For any  $F \in \mathbf{W}(\varrho)$  and  $p \in \mathcal{C}_G^k$  the maximal Lyapunov exponent  $\chi(p; G)$  satisfies*

$$\chi(p; G) \leq C \frac{1}{2^k},$$

where  $C = C(G)$  is some constant independent of  $k$ .

*Proof.* Let  $i = q2^k + n$ ,  $n = 2^{j_1} + 2^{j_2} + \dots + 2^{j_m} < 2^k$  and  $p \in H_{k,G}(\Lambda_*^k(\mathcal{C}_*))$ . Denote

$$t \equiv \Lambda_{k,G}^{-1}(p) \in \mathcal{C}_{G_k}, \quad s \equiv \Lambda_{k,G}(G_k^q(t)) \in \Lambda_{k,G}(\mathcal{C}_{G_k}),$$

then

$$\begin{aligned} DG^i(p) &= DG^{n+q2^k}(p) = DG^n(G^{q2^k}(p)) \cdot DG^{q2^k}(p) \\ &= DG^n(\Lambda_{k,G} \circ G_k^q \circ \Lambda_{k,G}^{-1}(p)) \cdot \Lambda_{k,G} \cdot DG_k^q(\Lambda_{k,G}^{-1}(p)) \cdot \Lambda_{k,G}^{-1} \\ &= DG^n(s) \cdot \Lambda_{k,G} \cdot DG_k^q(t) \cdot \Lambda_{k,G}^{-1} \\ &= D\left(T_{[n],G} \circ \Lambda_{j_m,G}^{-1}\right)(s) \cdot \Lambda_{k,G} \cdot DG_k^q(t) \cdot \Lambda_{k,G}^{-1} \end{aligned}$$

where we have used the representation (40). According to Lemma 6.1

$$T_{[n]_{l+1}^m, G} \circ \Lambda_{j_m, G}^{-1}(s) \in \mathcal{E}_1 \cup \mathcal{E}_3.$$

Denote  $B_k$  - an upper bound on the derivative norm of  $G_k$  on its invariant set  $H_{G_k}(\mathcal{C}_*)$ . Then

$$(71) \quad \|DG^i(p)\| \leq \left(\frac{A}{|\lambda_-|}\right)^m \left(\frac{|\lambda_-|}{\mu_-}\right)^{j_m} B_k^q \left(\frac{|\lambda_-|}{\mu_-}\right)^k.$$

Finally,

$$\begin{aligned} \chi(p; G) &= \overline{\lim}_{i \rightarrow \infty} \frac{1}{i} \log [\|DG^i(p)\|] \leq \lim_{i \rightarrow \infty} \frac{1}{i} \log \left[ \left(\frac{A}{|\lambda_-|}\right)^m B_k^q \left(\frac{|\lambda_-|}{\mu_-}\right)^{j_m+k} \right] \\ &\leq \lim_{i \rightarrow \infty} \left\{ \frac{k}{i} \log \left[ A \left(\frac{|\lambda_-|}{\mu_-^2}\right) \right] + \frac{q}{i} \log B_k \right\} \leq \frac{1}{2^k} \log B_k. \end{aligned}$$

□

Clearly, the above result implies stability of the limit set:

**Corollary 6.9.** *For any  $F \in \mathbf{W}(\varrho)$  and  $p \in \mathcal{C}_G^\infty$  the maximal Lyapunov exponent  $\chi(p; G)$  is equal to zero.*

## 7. THE STABLE SET AS A CANTOR SET

In this Section we will sketch the construction of the stable set using the method of presentation functions. The construction of this Section is almost identical to that of (de Carvalho *et al* 2005), and we will therefore omit many details. In fact, we will attempt to use the notation similar to that of (de Carvalho *et al* 2005).

Given  $F \in \mathbf{W}(\varrho)$  define the two *presentation functions*

$$\psi_0^F = \Lambda_F \quad \text{and} \quad \psi_1^F = F \circ \psi_0^F,$$

and

$$(72) \quad \Psi_0^{F,1} \equiv \psi_0^F \quad \text{and} \quad \Psi_1^{F,1} \equiv \psi_1^F,$$

$$(73) \quad \Psi_w^{F,n} \equiv \psi_{w_1}^F \circ \dots \circ \psi_{w_n}^{F^n}, \quad w = (w_1, \dots, w_n) \in \{0, 1\}^n.$$

**Lemma 7.1.** *For every  $F \in \mathbf{W}(\varrho)$  there exists a simply connected closed set  $B_F$  such that  $B_0^1(F) \equiv \psi_0^F(B_F) \subset B_F$  and  $B_1^1(F) \equiv \psi_1^F(B_F) \subset B_F$  are disjoint,  $F(B_1^1(F)) \cap B_0^1 \neq \emptyset$ , and*

$$(74) \quad \max\{\|D\psi_0^F\|_{B_F}, \|D\psi_1^F\|_{B_F}\} \leq \vartheta, \quad \vartheta = 0.272.$$

*Proof.* First, we verify the following on the computer:

$$\psi_0^F(\tilde{B}) \subset \tilde{B}, \quad \psi_1^F(\tilde{B}) \subset \tilde{B} \quad \text{and} \quad F(\psi_1^F(\tilde{B})) \cap \psi_0^F(\tilde{B}) \neq \emptyset$$

for all  $F \in \mathbf{W}(\varrho)$ , where

$$\tilde{B} = \{(x, u) \in \mathbb{R}^2 : \frac{(x - 0.47)^2}{0.82^2} + \frac{(u + 0.04)^2}{0.301398806^2} \leq 1\}.$$

We also check that the sets  $\psi_0^F(\tilde{B}) \subset \tilde{B}$  and  $\psi_1^F(\tilde{B}) \subset \tilde{B}$  are disjoint.

Second, we verify that the boundary of the ellipse  $\hat{B} \subset \tilde{B}$ ,

$$\hat{B} = \{(x, u) \in \mathbb{R}^2 : \frac{(x - 0.47)^2}{0.53^2} + \frac{u^2}{0.002370226^2} \leq 1\},$$

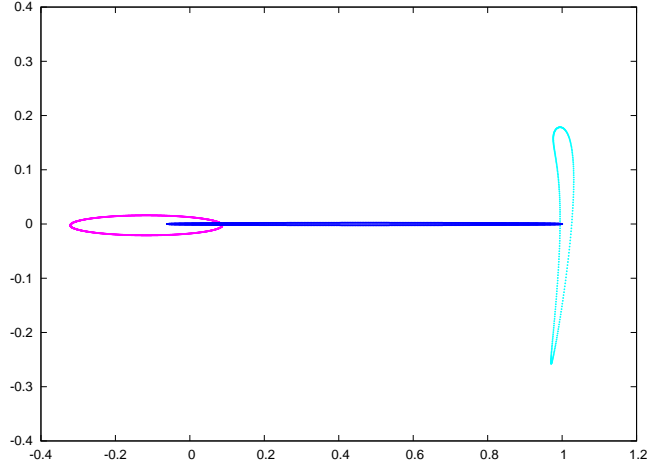


FIGURE 5. Sets  $\psi_0^F(\tilde{B})$  (magenta),  $\psi_1^F(\tilde{B})$  (cyan) and  $\hat{B}$  (blue).

intersects each of  $\psi_0^F(\tilde{B})$ ,  $\psi_1^F(\tilde{B})$  along a single arc. Therefore, the set

$$B \equiv \psi_0^F(\tilde{B}) \cap \psi_1^F(\tilde{B}) \cap \hat{B}$$

(see Figure 7) is simply connected, and satisfies the claim.  $\square$

Set  $B_0^1(F) = \psi_0^F(B_F)$ ,  $B_1^1(F) = \psi_1^F(B_F)$ , and define “pieces”

$$B_w^n(F) = \Psi_w^{F,n}(B_F), \quad w \in \{0, 1\}^n.$$

One can view  $\{0, 1\}^n$  as an additive group of residues *mod*  $2^n$  via an identification

$$w \rightarrow \sum_{k=0}^{n-1} w_{k+1} 2^k.$$

Let  $p : \{0, 1\}^n \rightarrow \{0, 1\}^n$ , be the operation of adding 1 in this group. The following Lemma has been proved in (de Carvalho *et al* 2005), and its proof holds in our case of area-preserving maps word by word:

**Lemma 7.2.**

1) *The above families of pieces are nested:*

$$B_{wv}^n \subset B_w^{n-1}, \quad w \in \{0, 1\}^{n-1}, \quad v \in \{0, 1\}.$$

2) *The pieces  $B_w^n$ ,  $w \in \{0, 1\}^n$  are pairwise disjoint.*

3)  *$F$  permutes the pieces as follows:  $F(B_w^n) = B_{p(w)}^n$  unless  $p(w) = 0^n$ . If  $p(w) = 0^n$ , then  $F(B_w^n) \cap B_{0^n}^n \neq \emptyset$ .*

4)  $\text{diam}(B_w^n) \leq \text{const } \vartheta^n$ .

5)  $C_d^H(\mathcal{C}_F^\infty) \leq -\log(2)/\log(\vartheta) < 0.5324$ , where

$$(75) \quad \mathcal{C}_F^\infty \equiv \bigcap_{n=1}^{\infty} \bigcup_{w \in \{0, 1\}^n} B_w^n.$$

Since the set  $\tilde{B}$  from Lemma 7.1 contains  $(0, 0)$ , so does each piece  $B_{0^n}^n$ . It follows from part 3) of Lemma 7.2 that the set  $\bigcup_{w \in \{0,1\}^n} B_w^n$  contains iterates  $G^i((0, 0))$  up to order  $2^n$ . Therefore, the Cantor set  $\mathcal{C}_F^\infty$  is the closure of the orbit of zero, and is equal to  $\mathcal{C}_G^\infty \cup F(\mathcal{C}_G^\infty) \cup F(F(\mathcal{C}_G^\infty))$ .

Recall the definition (28) of the dyadic group. Lemma 7.2 implies the following:

**Corollary 7.3.** *The restriction  $F|_{\mathcal{C}_F^\infty}$  is homeomorphic to  $p : \{0, 1\}^\infty \rightarrow \{0, 1\}^\infty$  via  $h : \{0, 1\}^\infty \rightarrow \mathcal{C}_F^\infty$  defined as*

$$h(w) = \bigcap_{n=1}^{\infty} B_{w_1 w_2 \dots w_n}^n.$$

## 8. “WEAK” RIGIDITY

In this Section we will demonstrate that the map  $\mathcal{H}_G$  is bi-Lipschitz for a subset of infinitely renormalizable maps.

**Proposition 8.1.** *There exist  $\varrho > 0$  and  $\omega$ ,*

$$\omega = \min \left\{ \frac{\mu_*}{|\lambda_*|}, \frac{b}{A} \right\},$$

*such that for all  $F \in \mathbf{W}_\omega(\varrho)$  the transformation  $\mathcal{H}_G$  is bi-Lipschitz with a constant  $\mathcal{L} = \mathcal{L}(\varrho)$ , that satisfies  $\mathcal{L}(\varrho) \xrightarrow{\varrho \rightarrow 0} 1$ .*

*Proof.* ‘Let  $i = 2^{j_1} + \dots + 2^{j_m}$  and  $\hat{i} = 2^{\hat{j}_1} + \dots + 2^{\hat{j}_n}$ ,  $i \neq \hat{i}$ , be arbitrary but fixed. Let  $\{p_{k,i}^*\}_{k=\max(j_m, \hat{j}_n)}^\infty$  and  $\{s_{k,\hat{i}}^*\}_{k=\max(j_m, \hat{j}_n)}^\infty$  be any two sequences of points that satisfy:  $p_{k,i}^* \in \mathcal{C}_{**}^{k,i}$  and  $s_{k,\hat{i}}^* \in \mathcal{C}_{**}^{k,\hat{i}}$ ,  $p_{k,i}^* \neq s_{k,\hat{i}}^*$ .

We would like to show, that there exist  $\varrho > 0$ ,  $\omega < \nu$  and  $\mathcal{L} = \mathcal{L}(\varrho)$ , such that if  $F \in \mathbf{W}_\omega(\varrho)$  then the distances

$$|{}_i H_{k,G}(p_{k,i}^*) - {}_{\hat{i}} H_{k,G}(s_{k,\hat{i}}^*)| \quad \text{and} \quad |p_{k,i}^* - s_{k,\hat{i}}^*|$$

are commensurate with a constant  $\mathcal{L}(\varrho)$ , independent of  $k$  and approaching 1 as  $\varrho \rightarrow 0$ .

Commensurability, together with convergence property (70) implies that the limit  $\mathcal{H}_G$  is a bi-Lipschitz transformation.

Define the following points:

$$\begin{aligned} p_{k,i} &= {}_i H_{k,G}(p_{k,i}^*), \\ s_{k,\hat{i}} &= {}_{\hat{i}} H_{k,G}(s_{k,\hat{i}}^*), \\ p_k^* &= G_*^{-i}(p_{k,i}^*) \equiv \Lambda_*^k(p^*), \\ s_k^* &= G_*^{-\hat{i}}(s_{k,\hat{i}}^*) \equiv \Lambda_*^k(s^*), \\ p_{k,i} &\equiv G^i(\Lambda_{k,G}(p)) \equiv G^i(\Lambda_{k,G}(H_{G_k}(p^*))), \\ s_{k,\hat{i}} &\equiv G^{\hat{i}}(\Lambda_{k,G}(s)) \equiv G^{\hat{i}}(\Lambda_{k,G}(H_{G_k}(s^*))), \end{aligned}$$

where the last four lines are understood as definitions of points  $p, s \in \mathcal{C}_{G_k}$  and  $p^*, s^* \in \mathcal{C}_*$ .

For any  $j < k$  and  $F \in \mathbf{W}_\omega(\varrho)$  there exists  $c'_7(\varrho)$  such that  $|\lambda_{G_j}| \leq |\lambda_*| + c'_7(\varrho)\omega^j$ , therefore

$$\begin{aligned}
 |\Lambda_{j,G}^{-1} \circ \Lambda_{k,G}(p) - \Lambda_*^{k-j}(p)| &= \left[ \prod_{n=j}^{k-1} (|\lambda_*| + c'_7(\varrho)\omega^n) - |\lambda_*|^{k-j} \right] |p| \\
 &= |\lambda_*|^{k-j} \left[ \exp \left\{ \sum_{n=j}^{k-1} \ln \left( 1 + c'_7(\varrho) \frac{\omega^n}{|\lambda_*|} \right) \right\} - 1 \right] |p| \\
 &= |\lambda_*|^{k-j} \left[ \exp \left\{ c''_7(\varrho) \omega^j \frac{1 - \omega^{k-j}}{1 - \omega} \right\} - 1 \right] |p| \\
 (76) \quad &\leq c_7(\varrho) |\lambda_*|^{k-j} \omega^j |p|,
 \end{aligned}$$

where  $c_7(\varrho)$  and  $c''_7(\varrho)$  are some constants. This, together with (53) implies the following bound for any  $p^*$  in  $\mathcal{C}_*$  and  $p = H_{G_k}(p^*)$  and all  $j < k$

$$\begin{aligned}
 |\Lambda_{j,G}^{-1}(\Lambda_{k,G}(p)) - \Lambda_*^{k-j}(p^*)| &\leq |\lambda_*|^{k-j} |p - p^*| + c_7(\varrho) \omega^j |\lambda_*|^{k-j} |p| \\
 &\leq C(\varrho) |\lambda_*|^{k-j} \omega^k + c_8(\varrho) \omega^j |\lambda_*|^{k-j},
 \end{aligned}$$

where  $c_8(\varrho) = c_7(\varrho) \text{diam}(\{C_G \cup (0, 0)\})$ .

Next, suppose that  $q$  is the smallest integer such that  $j_q \neq \hat{j}_q$  and  $j_l = \hat{j}_l$ ,  $l < q$ . For definitiveness, suppose  $\hat{j}_q > j_q$ . We expand, as before,

$$(77) \quad p_{k,i} = T_{[i]_1^{q-1}, G} \circ \left( \Lambda_{j_{q-1}, G}^{-1} \circ \Lambda_{j_q, G} \right) \circ \left[ G_{j_q} \circ T_{[i]_{q+1}^m, G} \circ \Lambda_{j_m, G}^{-1} \circ \Lambda_{k, G} \right] (p),$$

and similarly for  $G_*^i$ . Our immediate goal will be to show that  $|p_{k,i} - s_{k,i}|$  and  $|p_{k,i}^* - s_{k,i}^*|$  are commensurate. To this end we will show that the distances between the images of points  $p, s$  and  $p^*, s^*$  under the consecutive application of the three maps  $T_{[i]_1^{q-1}, G}$ ,  $(\dots)$  and  $\{\dots\}$  in (77) stay commensurate. We will perform this in three steps.

Step (1). Both

$$\bar{p}_q \equiv \left[ G_{j_q} \circ T_{[i]_{q+1}^m, G} \circ \Lambda_{j_m, G}^{-1} \circ \Lambda_{k, G} \right] (p) \quad \text{and} \quad \bar{p}_q^* \equiv \left[ G_* \circ T_{[i]_{q+1}^m} \circ \Lambda_*^{k-j_m} \right] (p^*)$$

lie in  $\mathcal{E}_2 \cup \mathcal{E}_4$ . We use (77) in the following bound

$$\begin{aligned}
 |\bar{p}_q - \bar{p}_q^*| &\leq \left| G_* \circ T_{[i]_{q+1}^m} \circ \Lambda_{j_m, G}^{-1} \circ \Lambda_{k, G}(p) - G_* \circ T_{[i]_{q+1}^m} \circ \Lambda_*^{k-j_m}(p^*) \right| \\
 &+ \left| G_{j_q} \circ T_{[i]_{q+1}^m, G} \circ \Lambda_{j_m, G}^{-1} \circ \Lambda_{k, G}(p) - G_* \circ T_{[i]_{q+1}^m} \circ \Lambda_{j_m, G}^{-1} \circ \Lambda_{k, G}(p) \right| \\
 &\leq \|D\mathbf{G}_*\|_{\mathcal{E}_1 \cup \mathcal{E}_3} A^{m-q} |\lambda_-|^{j_m - j_q - (m-q)} \times \\
 &\times \left[ C(\varrho) |\lambda_*|^{k-j_m} \omega^k + c_8(\varrho) \omega^{j_m} |\lambda_*|^{k-j_m} \right] + c_9(\varrho) \omega^{j_q} \\
 (78) \quad &\leq c_{10}(\varrho) \omega^{j_q}.
 \end{aligned}$$

Similarly,  $\bar{s}_q$  and  $\bar{s}_q^*$  are in  $\mathcal{E}_2 \cup \mathcal{E}_4$ , and  $|\bar{s}_q - \bar{s}_q^*| \leq c_{10}(\varrho) \omega^{j_q}$ .

Step (2). Next, denote,

$$\begin{aligned}
 \hat{p}_q &\equiv \Lambda_{j_{q-1}, G}^{-1} \circ \Lambda_{j_q, G}(\bar{p}_q), & \hat{p}_q^* &\equiv \Lambda_*^{j_q - j_{q-1}}(\bar{p}_q^*), & \tilde{p}_q &\equiv \Lambda_*^{j_q - j_{q-1}}(\bar{p}_q), \\
 \hat{s}_q &\equiv \Lambda_{j_{q-1}, G}^{-1} \circ \Lambda_{\hat{j}_q, G}(\bar{s}_q), & \hat{s}_q^* &\equiv \Lambda_*^{\hat{j}_q - j_{q-1}}(\bar{s}_q^*), & \tilde{s}_q &\equiv \Lambda_*^{\hat{j}_q - j_{q-1}}(\bar{s}_q).
 \end{aligned}$$

According to Lemma 6.2 the sets  $\mathcal{E}_2$  and  $\mathcal{E}_4$  are horizontally separated from their rescalings, therefore,

$$(79) \quad |\bar{p}_q - \Lambda_{j_q, G}^{-1} \circ \Lambda_{\hat{j}_q, G}(\bar{s}_q)| \geq \inf_{p \in \mathcal{E}_2 \cup \mathcal{E}_4} |\mathcal{P}_x p| - |\lambda_-| \sup_{p \in \mathcal{E}_2 \cup \mathcal{E}_4} |\mathcal{P}_x p| \equiv \delta_1.$$

Clearly, there is also a constant  $\delta_2$ , such that

$$\delta_2 > |\bar{p}_q - \Lambda_{j_q, G}^{-1} \circ \Lambda_{\hat{j}_q, G}(\bar{s}_q)|.$$

One can now use a computation similar to (76), to show that there exists a  $c_{11}(\varrho)$ , such that

$$|\hat{p}_q - \hat{s}_q| = \left| \Lambda_{j_{q-1}, G}^{-1} \circ \Lambda_{j_q, G} \left( \bar{p}_q - \Lambda_{j_q, G}^{-1} \circ \Lambda_{\hat{j}_q, G}(\bar{s}_q) \right) \right|$$

satisfies

$$(80) \quad \delta_2(1 + c_{11}(\varrho) \omega^{j_{q-1}}) |\lambda_*|^{j_q - j_{q-1}} \geq |\hat{p}_q - \hat{s}_q| \geq \delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}}) |\lambda_*|^{j_q - j_{q-1}},$$

and similarly for  $|\hat{p}_q^* - \hat{s}_q^*|$ . We use estimates (76) and (78) to compare these two distances:

$$\begin{aligned} |\hat{p}_q - \hat{s}_q| &\leq |\hat{p}_q^* - \hat{s}_q^*| + |\hat{p}_q - \tilde{p}_q| + |\tilde{p}_q - \hat{p}_q^*| + |\hat{s}_q - \tilde{s}_q| + |\tilde{s}_q - \hat{s}_q^*| \\ &\leq |\hat{p}_q^* - \hat{s}_q^*| + c_7(\varrho) |\lambda_*|^{j_q - j_{q-1}} \omega^{j_{q-1}} |\bar{p}_q| + |\lambda_*|^{j_q - j_{q-1}} |\bar{p}_q^* - \bar{p}_q| \\ &\quad + c_7(\varrho) |\lambda_*|^{j_q - j_{q-1}} \omega^{j_{q-1}} |\bar{s}_q| + |\lambda_*|^{j_q - j_{q-1}} |\bar{s}_q^* - \bar{s}_q| \\ &\leq |\hat{p}_q^* - \hat{s}_q^*| + c_7(\varrho) |\lambda_*|^{j_q - j_{q-1}} \omega^{j_{q-1}} (|\bar{p}_q| + |\bar{s}_q|) \\ &\quad + 2c_{10}(\varrho) |\lambda_*|^{j_q - j_{q-1}} \omega^{j_q} \end{aligned}$$

Therefore, there exists a  $c_{12}(\varrho)$ , such that

$$|\hat{p}_q - \hat{s}_q| \leq |\hat{p}_q^* - \hat{s}_q^*| + c_{12}(\varrho) |\lambda_*|^{j_q - j_{q-1}} \omega^{j_{q-1}},$$

and  $\varrho$  can be chosen sufficiently small, so that, for instance,

$$c_{12}(\varrho) < 1/2 \cdot \delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}}),$$

then

$$\frac{|\hat{p}_q - \hat{s}_q|}{|\hat{p}_q^* - \hat{s}_q^*|} \leq 1 + \frac{c_{12}(\varrho) |\lambda_*|^{j_q - j_{q-1}} \omega^{j_{q-1}}}{\delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}}) |\lambda_*|^{j_q - j_{q-1}}} \leq \frac{3}{2}.$$

One can use a similar argument to show that  $\varrho$  can be chosen sufficiently small, so that  $|\hat{p}_q^* - \hat{s}_q^*|/|\hat{p}_q - \hat{s}_q|$  is also bounded from above by a constant. In particular, if  $q = 1$ , then  $|\hat{p}_q^* - \hat{s}_q^*| = |p_{k,i}^* - s_{k,\hat{i}}^*|$  and  $|\hat{p}_q - \hat{s}_q| = |p_{k,i} - s_{k,\hat{i}}|$ , and

$$|{}_i H_{k,G}(p_{k,i}^*) - {}_{\hat{i}} H_{k,G}(s_{k,\hat{i}}^*)| = |p_{k,i} - s_{k,\hat{i}}| \asymp |p_{k,i}^* - s_{k,\hat{i}}^*|.$$

Step (3). Suppose that  $q > 1$ . We will now demonstrate that

$$(81) \quad |T_{[\hat{i}]_1^{q-1}}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q)| \asymp |p_{k,i} - s_{k,\hat{i}}|.$$

Denote for brevity

$$\text{Diff}_l = \left( T_{[i]_{j_{l+1}}^{j_{q-1}}, G}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}, G}(\hat{s}_q) \right) - \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right).$$

First, recall that  $G_* - G_{j_m} = O(\omega^{j_m})$ , and so is the difference of derivatives at any point contained in a compact subset of the domain  $\mathcal{D}_3$ . A straightforward calculation gives for any such point  $t$ :

$$(82) \quad DT_{j_m, j_{m-1}, G}(t) \cdot DT_{j_m - j_{m-1}}(t)^{-1} = \mathbf{O}(\omega^{j_{m-1}}),$$

$$(83) \quad DT_{j_1, j_0, G}(t) \cdot DT_{j_1}(t)^{-1} = \mathbf{O}(\varrho),$$

where  $\mathbf{O}(\delta)$  signifies a  $2 \times 2$  matrix whose elements are  $O(\delta)$ .

Next,

$$\begin{aligned} \text{Diff}_{q-2} &= \left( DT_{[i]_{j_{q-1}}^{j_{q-1}}, G}(t_{q-2}^G) - DT_{[i]_{j_{q-1}}^{j_{q-1}}}(t_{q-2}^*) \right) \cdot (\hat{p}_q - \hat{s}_q) \\ &= \left( DT_{j_{q-1}, j_{q-2}, G}(t_{q-2}^G) - DT_{j_{q-1} - j_{q-2}}(t_{q-2}^*) \right) \cdot (\hat{p}_q - \hat{s}_q), \end{aligned}$$

where the derivatives are evaluated at some points  $t_{q-2}^G$  and  $t_{q-2}^*$  on the line segment between  $\hat{p}_q$  and  $\hat{s}_q$  (Mean Value Theorem). Again, it is straightforward to demonstrate that  $t_{q-2}^G - t_{q-2}^* = O(\omega^{j_{q-2}})$ , therefore

$$\begin{aligned} \text{Diff}_{q-2} &= \left( DT_{[i]_{j_{q-1}}^{j_{q-1}}, G}(t_{q-2}^G) \cdot DT_{[i]_{j_{q-1}}^{j_{q-1}}}(t_{q-2}^*)^{-1} - I \right) \cdot DT_{[i]_{j_{q-1}}^{j_{q-1}}}(t_{q-2}^*) \cdot (\hat{p}_q - \hat{s}_q) \\ &= \mathbf{O}(\omega^{j_{q-2}}) \cdot DT_{[i]_{j_{q-1}}^{j_{q-1}}}(t_{q-2}^*) \cdot (\hat{p}_q - \hat{s}_q) \\ (84) \quad &= \mathbf{O}(\omega^{j_{q-2}}) \cdot \left( T_{[i]_{j_{q-1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{q-1}}^{j_{q-1}}}(\hat{s}_q) \right). \end{aligned}$$

Assume, that for some  $2 \leq l < q - 2$ ,

$$\text{Diff}_l = \mathbf{O}(\omega^{j_l}) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right).$$

Then

$$\begin{aligned} \text{Diff}_{l-1} &= \left( T_{j_l, j_{l-1}, G} \circ T_{[i]_{j_{l+1}}^{j_{q-1}}, G}(\hat{p}_q) - T_{j_l, j_{l-1}, G} \circ T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}, G}(\hat{s}_q) \right) \\ &\quad - \left( T_{j_l, j_{l-1}} \circ T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{j_l, j_{l-1}} \circ T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \\ &= DT_{j_l, j_{l-1}, G}(\tilde{t}_{l-1}^G) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}, G}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}, G}(\hat{s}_q) \right) \\ &\quad - DT_{j_l, j_{l-1}}(\tilde{t}_{l-1}^*) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \\ &= \left\{ DT_{j_l, j_{l-1}, G}(\tilde{t}_{l-1}^G) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \right. \\ &\quad \left. - DT_{j_l, j_{l-1}}(\tilde{t}_{l-1}^*) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \right\} \\ &\quad + \left[ DT_{j_l, j_{l-1}, G}(\tilde{t}_{l-1}^G) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}, G}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}, G}(\hat{s}_q) \right) \right. \\ &\quad \left. - DT_{j_l, j_{l-1}, G}(\tilde{t}_{l-1}^*) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \right], \end{aligned}$$

where  $\tilde{t}_{l-1}^G$  is some point on the line segment connecting  $T_{[i]_{j_{l+1}}^{j_{q-1}}, G}(\hat{p}_q)$  and  $T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}, G}(\hat{s}_q)$ , while  $\tilde{t}_{l-1}^*$  is a point on the line segment between  $T_{[i]_{j_l}^{j_{q-1}}}(\hat{p}_q)$  and  $T_{[\hat{i}]_{j_l}^{j_{q-1}}}(\hat{s}_q)$ .

We treat expression  $\{\dots\}$  and  $[\dots]$  separately. The first one is worked out similarly to (84):

$$\begin{aligned} \{\dots\} &= \mathbf{O}(\omega^{j_{l-1}}) \cdot DT_{j_l, j_{l-1}}(\tilde{t}_{l-1}^*) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \\ &= \mathbf{O}(\omega^{j_{l-1}}) \cdot \left( T_{[i]_{j_l}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_l}^{j_{q-1}}}(\hat{s}_q) \right). \end{aligned}$$

$$\begin{aligned} [\dots] &= DT_{j_l, j_{l-1}, G}(\tilde{t}_{l-1}^G) \cdot \text{Diff}_1 = (I + \mathbf{O}(\omega^{j_{l-1}})) \cdot DT_{j_l, j_{l-1}}(\tilde{t}_{l-1}^*) \cdot \text{Diff}_1 \\ &= (I + \mathbf{O}(\omega^{j_{l-1}})) \cdot DT_{j_l, j_{l-1}}(\tilde{t}_{l-1}^*) \cdot \mathbf{O}(\omega^{j_l}) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \\ &= (I + \mathbf{O}(\omega^{j_{l-1}})) \cdot \mathbf{O}(\omega^{j_l}) \cdot DT_{j_l, j_{l-1}}(\tilde{t}_{l-1}^*) \cdot \left( T_{[i]_{j_{l+1}}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_{l+1}}^{j_{q-1}}}(\hat{s}_q) \right) \\ &= (I + \mathbf{O}(\omega^{j_{l-1}})) \cdot \mathbf{O}(\omega^{j_l}) \cdot \left( T_{[i]_{j_l}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_l}^{j_{q-1}}}(\hat{s}_q) \right). \end{aligned}$$

Therefore,

$$\text{Diff}_{l-1} = \mathbf{O}(\omega^{j_{l-1}}) \cdot \left( T_{[i]_{j_l}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_l}^{j_{q-1}}}(\hat{s}_q) \right),$$

which completes the induction. In particular, when  $l = 1$ , we obtain

$$\left( T_{[i]_{j_1}^{j_{q-1}}, G}(\hat{p}_q) - T_{[\hat{i}]_{j_1}^{j_{q-1}}, G}(\hat{s}_q) \right) = (I + \mathbf{O}(\varrho)) \cdot \left( T_{[i]_{j_1}^{j_{q-1}}}(\hat{p}_q) - T_{[\hat{i}]_{j_1}^{j_{q-1}}}(\hat{s}_q) \right),$$

where we have used (83), or

$$(85) \quad |T_{[i]_1^{q-1}, G}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}, G}(\hat{s}_q)| \leq (1 + c_{13}(\varrho)) |T_{[i]_1^{q-1}}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q)|.$$

for some “constant”  $c_{13}$ ,  $\lim_{\varrho \rightarrow 0} c_{13}(\varrho) = 0$ .

On the other hand, if we interchange the roles of  $G$  and  $G_*$  in the computation above, we get

$$|T_{[i]_1^{q-1}}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q)| \leq (1 + c_{13}(\varrho)) |T_{[i]_1^{q-1}, G}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}, G}(\hat{s}_q)|.$$

Step (4). Finally, we demonstrate that

$$|p_{k,i} - s_{k,\hat{i}}| = |T_{[i]_1^{q-1}, G}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}, G}(\hat{s}_q)|$$

and

$$|p_{k,i}^* - s_{k,\hat{i}}^*| = |T_{[i]_1^{q-1}}(\hat{p}_q^*) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q^*)|$$

are commensurate.

We will compare

$$I_1 \equiv |T_{[i]_1^{q-1}}(\hat{p}_q^*) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q^*)|$$

and

$$I_2 \equiv |T_{[i]_1^{q-1}}(\hat{p}_q) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q)|.$$

Since  $T_{[i]_1^{q-1}} = T_{[\hat{i}]_1^{q-1}}$ ,

$$(86) \quad b^{q-1} \mu_*^{j_{q-1} - (q-1)} |\hat{s}_q^* - \hat{p}_q^*| \leq I_1 \leq A^{q-1} |\lambda_*|^{j_{q-1} - (q-1)} |\hat{s}_q^* - \hat{p}_q^*|,$$

and

$$\begin{aligned}
I_2 &\leq I_1 + |T_{[i]_1^{q-1}}(\hat{p}_q^*) - T_{[i]_1^{q-1}}(\hat{p}_q)| + |T_{[\hat{i}]_1^{q-1}}(\hat{s}_q^*) - T_{[\hat{i}]_1^{q-1}}(\hat{s}_q)| \\
&\leq I_1 + A^{q-1} |\lambda_*|^{j_{q-1}-(q-1)} (|\hat{p}_q^* - \hat{p}_q| + |\hat{s}_q^* - \hat{s}_q|) \\
&\leq I_1 + c_{12}(\varrho) A^{q-1} |\lambda_*|^{j_{q-1}-(q-1)} \omega^{j_{q-1}}.
\end{aligned}$$

These two estimates put together with the estimate (80) result in the following bound:

$$\begin{aligned}
\frac{I_2}{I_1} &\leq 1 + c_{12}(\varrho) \frac{A^{q-1} |\lambda_*|^{j_{q-1}-(q-1)} \omega^{j_{q-1}}}{b^{q-1} \mu_*^{j_{q-1}-(q-1)} |\hat{p}_q^* - \hat{s}_q^*|} \omega^{j_{q-1}} \\
&\leq 1 + \frac{c_{12}(\varrho)}{\delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}})} \frac{A^{q-1} |\lambda_*|^{j_{q-1}-(q-1)}}{b^{q-1} \mu_*^{j_{q-1}-(q-1)} |\lambda_*|^{j_{q-1}-j_{q-1}}} \omega^{j_{q-1}} \\
&\leq 1 + \frac{c_{12}(\varrho)}{\delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}})} \left( \frac{A \mu_*}{b |\lambda_*|} \right)^{q-1} \left[ \frac{|\lambda_*|}{\mu_*} \right]^{j_{q-1}} \omega^{j_{q-1}} \\
&\leq \begin{cases} 1 + \frac{c_{12}(\varrho)}{\delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}})} \left( \frac{A \mu_*}{b |\lambda_*|} \right)^{j_{q-1}} \left[ \frac{|\lambda_*|}{\mu_*} \right]^{j_{q-1}} \omega^{j_{q-1}}, & \frac{A \mu_*}{b |\lambda_*|} > 1 \\ 1 + \frac{c_{12}(\varrho)}{\delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}})} \left[ \frac{|\lambda_*|}{\mu_*} \right]^{j_{q-1}} \omega^{j_{q-1}}, & \frac{A \mu_*}{b |\lambda_*|} \leq 1 \end{cases} \\
(87) \quad &\leq 1 + \frac{c_{12}(\varrho)}{\delta_1(1 - c_{11}(\varrho) \omega^{j_{q-1}})} \max \left\{ \frac{|\lambda_*|}{\mu_*}, \frac{A}{b} \right\}^{j_{q-1}} \omega^{j_{q-1}}.
\end{aligned}$$

and similarly for  $I_1/I_2$ . Therefore, if

$$\omega \leq \min \left\{ \frac{\mu_*}{|\lambda_*|}, \frac{b}{A} \right\},$$

then

$$(88) \quad |{}_i H_{k,G}(p_{k,i}^*) - {}_{\hat{i}} H_{k,G}(s_{k,\hat{i}}^*)| = |p_{k,i} - s_{k,\hat{i}}| \asymp I_2 \asymp I_1 = |p_{k,i}^* - s_{k,\hat{i}}^*|.$$

□

**Remark 8.2.** We would like to emphasize that the commensurability property (88) holds only for  $i \neq \hat{i}$ , and therefore does not imply that the hyperbolic sets  $\mathcal{C}_G^k$  and  $\mathcal{C}_*^k$  are bi-Lipschitz conjugate. In case  $i = \hat{i}$  a positive lower bound (79) does not exist, which would invalidate the arguments that follow.

## 9. SOME CONCLUDING REMARKS

We have demonstrated that the Hausdorff dimension of the stable set for the maps  $F$  in the subset  $\mathbf{W}_\omega(\varrho)$  of the infinitely renormalizable maps is independent of  $F$ , and that the stable dynamics for two infinitely renormalizable maps in  $\mathbf{W}_\omega(\varrho)$  is bi-Lipschitz-conjugate. This is quite weaker than the corresponding result about the invariance of the Hausdorff dimension of the Feigenbaum attractor for all infinitely renormalizable unimodal maps (see (Paluba 1989, Rand 1988, McMullen 1996, de Melo and Pinto 1999)). On the other hand, it does demonstrate that one should expect at least some kind of rigidity of invariant sets for infinitely renormalizable maps in conservative dynamics — rigidity which was absent in dissipative maps (see (de Carvalho *et al* 2005)).

Our proof of the bi-Lipschitz property of the conjugacy between stable sets  $\mathcal{C}_G^\infty$  and  $\mathcal{C}_G^\infty$  balances two phenomena that, in a sense, work against each other: convergence of renormalizations of maps  $G \in \mathbf{W}(\varrho)$  versus the fact that the rates of contraction of distances in different directions by maps  $\Lambda_F$  and  $\Lambda_F \circ F$  are essentially different. A careful look at the proof shows that the bi-Lipschitz property is achieved if the convergence rate  $\nu$  is sufficiently small to “counteract” the relative size of contractions. However, this is not the case with the upper bound (15) on  $\nu$  at hand. Although this upper bound is by no means sharp, it does indicate that one might need to choose a submanifold  $\mathbf{W}_\omega(\varrho)$  of  $\mathbf{W}(\varrho)$  on which the convergence rate is smaller.

Another obvious issue for investigation is whether the bi-Lipschitz conjugacy of the stable sets extends to their neighborhood as a  $C^{1+\epsilon}$  map. Again, this is the case for the conjugacies between attractors of the unimodal maps (see (Rand 1988, McMullen 1996, de Melo and Pinto 1999)), and it is not for very dissipative maps where, as we have already mentioned, the regularity of the conjugacy of attractors for two maps  $F$  and  $\tilde{F}$  has a definite upper bound (1).

## 10. ACKNOWLEDGMENTS

The authors would like to thank Hans Koch for his many useful insights into period-doubling in area-preserving maps, as well as for his great help with understanding the original computer assisted proof (Eckmann *et al* 1984).

## REFERENCES

- J. J. Abad, H. Koch, Renormalization and periodic orbits for Hamiltonian flows, *Comm. Math. Phys.* **212** (2000) # 2 371–394.
- J. J. Abad, H. Koch and P. Wittwer, A renormalization group for Hamiltonians: numerical results, *Nonlinearity* **11** (1998) 1185–1194.
- G. Benettin et al, Universal properties in conservative dynamical systems, *Lettere al Nuovo Cimento* **28** (1980) 1–4.
- T. Bountis, Period doubling bifurcations and universality in conservative Systems, *Physica* **3D** (1981) 577–589.
- CAPD—Computer Assisted Proofs in Dynamics group, a C++ package for rigorous numerics, <http://capd.wsb-nlu.edu.pl>
- A. de Carvalho, M. Lyubich, M. Martens, Renormalization in the Hénon family, I: Universality but non-rigidity, *J. Stat. Phys.* **121** (2005) 611–669.
- P. Collet, J.-P. Eckmann and H. Koch, Period doubling bifurcations for families of maps on  $\mathbb{R}^n$ , *J. Stat. Phys.* **3D** (1980).
- P. Collet, J.-P. Eckmann and H. Koch, On universality for area-preserving maps of the plane, *Physica* **3D** (1981) 457–467.
- B. Derrida, Y. Pomeau, Feigenbaum’s ratios of two dimensional area preserving maps, *Phys. Lett.* **A80** (1980) 217–219.
- P. Duarte, Persistent homoclinic tangencies for conservative maps near identity, *Ergod. Th. & Dynam. Sys.* **20** (2000) 393–438.
- J.-P. Eckmann, H. Koch and P. Wittwer, Existence of a fixed point of the doubling transformation for area-preserving maps of the plane, *Phys. Rev. A* **26** (1982) # 1 720–722.
- J.-P. Eckmann, H. Koch and P. Wittwer, A Computer-Assisted Proof of Universality for Area-Preserving Maps, *Memoirs of the American Mathematical Society* **47** (1984), 1–121.
- H. Epstein, New proofs of the existence of the Feigenbaum functions, *Commun. Math. Phys.* **106** (1986) 395–426.
- H. Epstein, Fixed points of composition operators II, *Nonlinearity* **2** (1989) 305–310.

- D. F. Escande, F. Doveil, Renormalization method for computing the threshold of the large scale stochastic instability in two degree of freedom Hamiltonian systems, *J. Stat. Phys.* **26** (1981) 257–284.
- E. de Faria, *Proof of universality for critical circle mappings*, Thesis, CUNY, 1992.
- E. de Faria, Asymptotic rigidity of scaling ratios for critical circle mappings, *Ergodic Theory Dynam. Systems* **19** (1999), no. 4, 995–1035.
- M. J. Feigenbaum, Quantitative universality for a class of nonlinear transformations, *J. Stat. Phys.* **19** (1978) 25–52.
- M. J. Feigenbaum, Universal metric properties of non-linear transformations, *J. Stat. Phys.* **21** (1979) 669–706.
- D. Gaidashev, Renormalization of isoenergetically degenerate Hamiltonian flows and associated bifurcations of invariant tori, *Discrete Contin. Dyn. Syst.* **13** (2005), no. 1, 63–102.
- D. Gaidashev, Cylinder renormalization for Siegel disks and a constructive Measurable Riemann Mapping Theorem, *Nonlinearity* **20** (2007), no. 3, 713–742.
- D. Gaidashev, H. Koch, Renormalization and shearless invariant tori: numerical results, *Nonlinearity* **17** (2004), no. 5, 1713–1722.
- D. Gaidashev, H. Koch, Period doubling in area-preserving maps: an associated one-dimensional problem, e-print [math.DS/0811.2588](https://arxiv.org/abs/math.DS/0811.2588) at [Arxiv.org](https://arxiv.org/).
- D. Gaidashev, M. Yampolsky, Cylinder renormalization of Siegel disks, *Exp. Math.* **16:2** (2007).
- D. Gaidashev, T. Johnson, Dynamics of the Universal Area-Preserving Map Associated with Period Doubling: Hyperbolic Sets, *Nonlinearity* **22** 2487–2520.
- R. H. G. Helleman, Self-generated chaotic behavior in nonlinear mechanics, in "Fundamental problems in statistical mechanics", Ed. by E. G. D. Cohen, North-Holland, Amsterdam, p.165, (1980).
- A. Katok, B. Hasselblat, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge (1995).
- K. Khanin, J. Lopes Dias, J. Marklof, Multidimensional continued fractions, dynamic renormalization and KAM theory, *Comm. Math. Phys.*, **270** (2007), no. 1, 197–231.
- H. Koch, On the renormalization of Hamiltonian flows, and critical invariant tori, *Discrete Contin. Dyn. Syst.* **8** (2002), 633–646.
- H. Koch, A renormalization group fixed point associated with the breakup of golden invariant tori, *Discrete Contin. Dyn. Syst.* **11** (2004), no. 4, 881–909.
- H. Koch, Existence of critical invariant tori, *Ergodic theory and dynamical systems* **28** 1879–94.
- S. Kocić, Renormalization of Hamiltonians for Diophantine frequency vectors and KAM tori, *Nonlinearity* **18** (2005) 2513–2544.
- H. Kokubu, D. Wilczak and P. Zgliczyński, Rigorous verification of cocoon bifurcations in the Michelson system, *Nonlinearity* **20** no. 9 (2007), 2147–2174.
- M. Lyubich, Feigenbaum-Coullet-Tresser universality and Milnor's hairness conjecture, *Annals of Mathematics* **149** (1999) 319–420.
- M. Lyubich, M. Martens, Renormalization in the Hénon family, II: Homoclinic tangle, preprint, [arXiv:mathDS/0804.0780](https://arxiv.org/abs/mathDS/0804.0780) (2008).
- R. S. MacKay, Renormalisation in area preserving maps, Thesis, Princeton (1982). World Scientific, London (1993).
- R. S. MacKay, Renormalization approach to invariant circles in area-preserving maps, *Physica D* **7** (1983) 283–300.
- A. Mehr and D.F. Escande, Destruction of KAM tori in Hamiltonian systems: link with the destabilization of nearby cycles and calculation of residues, *Physica D* **13** (1984) 302–338.
- W. de Melo, A. A. Pinto, Rigidity of  $C^2$  infinitely renormalizable unimodal maps, *Comm. Math. Phys.*, **208** (1999), 91–105.
- C. McMullen, *Renormalization and 3-manifolds which fiber over the circle. (Annals of Mathematics Studies, 142)*, Princeton University Press, Princeton, NJ, 1996.
- C. McMullen, Self-similarity of Siegel disks and Hausdorff dimension of Julia sets, *Acta Math.* **180** (1998), 247–292.
- W. Paluba, The Lipschitz condition for the conjugacies of Feigenbaum-like mappings, *Fund. Math.*, **132** (1989), 227–258.
- D. A. Rand, Global phase space universality, smooth conjugacies and renormalisation. I. The  $C^{1+\alpha}$  case, *Nonlinearity*, **1** (1988), 181–202.

- S. J. Shenker, L. P. Kadanoff, Critical behaviour of KAM surfaces. I Empirical results, *J. Stat. Phys.* **27** (1982) 631–656.
  - D. Sullivan, Bounds, quadratic differentials and renormalization conjectures, in: Mathematics into the Twenty-first Century, AMS Centennial Publications, Vol. II, Amer. Math. Soc., Providence, R.I. (1992) 417–466.
  - C. Tresser and P. Coullet, Itérations d’endomorphismes et groupe de renormalisation, *C. R. Acad. Sci. Paris* **287A** (1978), 577–580.
  - M. Yampolsky, Hyperbolicity of renormalization of critical circle maps, *Publ. Math. Inst. Hautes Etudes Sci.* **96** (2002), 1–41.
  - M. Yampolsky, Renormalization horseshoe for critical circle maps, *Commun. Math. Physics* **240** (2003), 75–96.
  - M. Yampolsky, Siegel disks and renormalization fixed points, *Holomorphic Dynamics and Renormalization (Fields Inst. Commun. vol 53)* (Providence, RI:AMS) pp. 377–93.
  - P. Zgliczyński, Computer assisted proof of chaos in the Rössler equations and in the Hénon map, *Nonlinearity* **10**, no. 1 (1997), 243–252.
  - P. Zgliczyński, Covering relations, cone conditions and the stable manifold theorem, *Journal of Differential Equations* **246** issue 5 (2009), 1774–1819.
  - P. Zgliczyński and M. Gidea 2004 Covering relations for multidimensional dynamical systems. *J. Differential Equations* **202** (2004), no. 1, 32–58.
- Programs available at <http://math.uu.se/~gaidash>

DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, BOX 480, 751 06 UPPSALA, SWEDEN  
*E-mail address:* [gaidash@math.uu.se](mailto:gaidash@math.uu.se), [tomas.johnson@math.uu.se](mailto:tomas.johnson@math.uu.se)

